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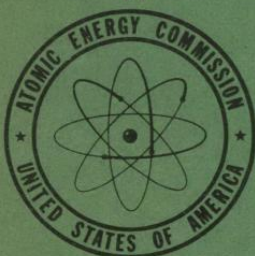
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TECHNICAL REPORTS

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Technical Report No. 1

DESIGN OF CYCLOTRON MAGNETS

By M. H. Foss

ABSTRACT

The study of cyclotron-magnet design presented here is directed toward attaining a large ratio of energy in the ion beam with respect to cost. Alternatively, the goal may be regarded as a smaller, more convenient apparatus. These two objectives are not completely incompatible; a design that is more economical than the usual is also smaller.

The best design of a cyclotron can be obtained only by considering the problem as a whole. The conditions necessary to maintain an ion beam, the conditions required for the operation of the oscillator, and a host of other problems influence the design of the magnet. It is assumed in this paper that the "other problems" can be satisfactorily solved, if the gap has a suitable geometry and the field a suitable space dependence.

By subdividing the problem, the design of a cyclotron magnet is treated in four parts, the pole tips, coils, yoke, and experimental methods.

The pole tips are treated in Sec. 1.1. The equation for the scalar potential of a mid-plane field which decreases linearly with increasing radius is given. The equipotentials of this field are such that they would be obtained only with a steel structure crossing the gap at its center. Since this space is required for other parts of the apparatus, the termination of the field at the center of the gap is given special treatment. The termination of the field outside the gap is treated by finding a series of two-dimensional solutions of Laplace's equation for various acceptable terminations of the field. That these fields can be approximately realized with steel pole tips is inferred from the position of certain equipotential surfaces or from the flux density at certain points in the field. The treatment is incomplete in the sense that no method of determining the required steel shape is developed, but a clear picture of the problems encountered, and the results to be expected, is brought out.

Section 1.2 is devoted to coil design. Small coils make a more convenient cyclotron and reduce the amount of steel required in the flux circuit. The arrangement of conductors and the cooling of such coils is studied. Equations are derived for the optimum dimensions of the coils.

In Sec. 1.3 the pole base, yoke, and optimum field are discussed. An equation for optimum flux density in the pole base is derived. Since the yoke can be designed easily by experimental means after the rest of the design is in approximately final form, the theoretical treatment of the yoke problem is not adequate for design purposes. It is shown, however, that area of horizontal yokes should be greater than that of vertical yokes. A final equation is presented for determining optimum field at which to operate a magnet. Like the other cost equations, it requires experimental data for its use.

In Sec. 1.4 a null method for determining the ratio of two quantities of flux is described. Only one absolute field measurement need be made on a model magnet. The remaining measurements can be made by the null method with good accuracy. Two methods for using the data are discussed. Many results were obtained which do not belong in a direct attack on the problem. Some of these results are useless and are omitted. Those which seemed worth recording are given in the appendixes.

Appendix I contains two tables, one for the determination of the smaller root of a quadratic equation and the other containing values of a function encountered in Sec. 1.1.

In Appendix II a method is developed for determining the field caused by infinitely permeable right cylindrical poles resting on yokes which are infinite planes. Any potential distribution on the pole side and yoke may be treated.

For its suggestive value, the potential equation for a two-dimensional pole edge and yoke excited by a line coil is given in Appendix III.

Two types of search coil for measuring the field at a point are discussed in Appendix IV. Also, tables of $P_m^1(\cos \theta)$ are given.

It is hoped that this work will be of assistance in designing future cyclotrons.

1.1 POLE TIPS

Introduction

An important and costly part of a large cyclotron is a magnet which is energized by direct current flowing in coils around the poles. The general features of such a magnet are shown in Fig. 1.1. In order to obtain an ion beam and to make it possible to develop the required r-f electric field in the gap, the magnet must have the features listed below. These are the basic assumptions on which the remaining work depends.

1. The magnetic field should be produced in a cylindrical region that contains none of the steel or copper used to produce the field. This region, called the "gap," should have a diameter about nine times as great as its axial dimension.
2. The magnetic field should be symmetric about a plane called the "mid-plane" of the gap.
3. The field in the mid-plane should depend on the radial distance from, but not the angular direction about, a point called the "center" of the gap.
4. The field should decrease with increasing distance from the center of the gap in such a way that the condition

$$n = -\frac{R}{H} \frac{dH}{dR} \ll \frac{1}{5} \quad (1)$$

where R = radius, and

H = magnetic field

is never violated except at radii greater than any which are to be used. The desired field is one which decreases linearly with increasing radius to the outside "edge" of the gap.

These conditions by no means fix the design of the magnet. The factors which control the rest of the design are convenience of operation, costs, and ignorance. Since the design of a magnet is complicated, the problem is divided into several parts. The pole tip will be discussed first.

In order to have an efficient magnetic circuit, one with low leakage, the pole tip, that part of the pole which projects above the coil, should be as short as possible. However, a pole tip of some length is required by those who use a cyclotron in order that they may have access to the gap.

The part of the pole tip which is essentially horizontal is usually called the "shim," and the contour of the remaining part of the pole tip is called the "side of the pole." The central part of the shim (to within a few half-gaps of the edge) can be treated as one problem, and the shim edge and side of the pole as another.

Any way that the desired field can be obtained in the central part of the gap is as good as any other, since all of this flux is useful. The equation for the equipotentials of the desired field, i.e., a field which decreases linearly with increasing radius, is given, and a possible termination of the shim in the center of the gap is given.

The termination of the shim at the outside edge of the gap and the shape of the side of the pole play very important roles in the performance of the magnet. Two factors are important, the ratio of the useful flux to the flux in the pole base and the ratio of the flux density in the gap to the flux density in the base of the pole.

The mathematical formulation of the shim-edge and pole-side problem is difficult because the flux density in the steel in this region varies from nominal to well above saturation, and the usual ways of avoiding the treatment of the field in the steel cannot be employed. This difficulty is avoided by simply not considering the field in the steel. A number of possible terminations of the field which satisfy Laplace's equation are studied. From the position of the equipotentials it is possible to estimate the position of the steel, and from the calculated flux density an assumption can be made as to whether steel can be used to produce the field. From this mass of calculated material it is possible to obtain good estimates of the performance which can be expected from a steel pole tip. These estimates are given in the subsection, Conclusions.

Field in Center of Gap

It is assumed that the desired magnetic field for a cyclotron is a symmetric field which, in the mid-plane, decreases linearly with increasing radius. These properties may be expressed by

$$V(+z_1) = -V(-z_1) \quad V|_{z=0} = 0$$

and

(2)

$$\left. \frac{H_z}{H_c} \right|_{z=0} = 1 - Ar$$

where H_c , A = constants

V = scalar magnetic potential

z, r, θ = cylindrical variables (the field does not depend on θ)

The coordinates and field are illustrated in Fig. 1.2.

The scalar potential of such a field is

$$\frac{V}{H_c} = Ar^2 f - z$$

where

$$f = \frac{1}{4} \left[3x \sqrt{x^2 + 1} + (1 - 2x) \sinh^{-1} x \right] \quad (3)$$

and

$$x = \frac{z}{r}$$

If the field is to fall off to $1 - \delta$ of its value at the center when the radius is (r_0) , then

$$A = \frac{\delta}{r_0^2}$$

and if steel had infinite permeability then it should be given the shape of the equipotential surface given by

$$V = -H_c S \quad (4)$$

where S , a constant, is one-half the nominal gap height. The equation for the shape of this equipotential surface is found by substituting Eq. 4 into Eq. 3.

$$\frac{\delta}{r_0} \frac{f}{x^2} z^2 - z + s = 0 \quad (5)$$

A method and tables for solving this equation are given in Appendix I, and a typical solution is given in Fig. 1.3.

Shape of Shim

Since the permeability of steel is not infinite, the shim will not have the shape of the equipotentials given in the previous paragraph. The analytic solution indicates that a spike of very small radius should cross the center of the gap. The steel in such a spike would completely saturate and also use space required for the ion source. It is thus not possible to keep the linear character of the field at the origin. Since the spike is small, it has little influence on the over-all performance of the magnet and can be adjusted at small cost, even on the final full-scale magnet. When the field shape without the spike is known, the dimensions of the spike can be estimated by the method indicated in the next paragraph.

The extent to which Eq. 5 gives the correct shim shape away from the center depends on the shape of the pole and the field at which it is to operate. The contour of the shim when the steel surface is not an equipotential is best determined experimentally. An approximate shim can be put on a pole which otherwise has its final form. The potential of the steel surface is then determined by direct measurement or by a numerical calculation from the field in the mid-plane of the gap. The surface can be reshaped according to Eq. 3 assuming that the potential of the steel surface at a given radius does not vary. Equation 3 can be replaced by the more convenient equation:

$$z \doteq -\left(1 + \frac{\delta}{r_1} r\right) \frac{V}{H_c} \quad (6)$$

to good approximation when the radius is greater than a few half-gaps.

Estimate of Spike Dimensions

Since the spikes will have a flux density well over saturation, the field in the gap may be taken approximately as the sum of the field which would be obtained with no spikes present, plus that of permanent magnets shaped like the spikes with their magnetization, M , given by

$$\frac{1}{4\pi} (B - H) = M = \frac{1}{4\pi} B_s \quad (7)$$

where B_s = intrinsic flux density of the steel. If the spikes are cylindrical, the field due to them may be treated approximately as that of point charges placed at their tips with strength

$$m = MA$$

where A = cross-sectional area of cylinders. In this case, the field due to the spikes may be determined with reasonable accuracy as the field due to the point charges at the tip of each spike and their images in the poles to which they are adjacent. This gives the equation

$$H_z \Big|_{z=0} = \frac{2m}{s^2} \left\{ \frac{(1 - h/s)}{\left[(1 - h/s)^2 + \left(\frac{r}{s} \right)^2 \right]^{3/2}} - \frac{(1 + h/s)}{\left[(1 + h/s)^2 + \left(\frac{r}{s} \right)^2 \right]^{3/2}} \right\} \quad (8)$$

where s = height of half-gap

h = height of spike

This function is given for a particular case in Fig. 1.4. If a field of this shape added to the field present without the spike is satisfactory, then the proper adjustment can be made by using a cylindrical spike with suitable area.

If the spike should require a radius of the same order of magnitude as its height, it may be desirable to have it tapered at the tip to reduce the demagnetizing field of the induced poles.

Shim Edge and Side of Pole, Introduction

The linear decrease of the mid-plane field with increasing radius can be produced by the potential given in Eq. 3 to within about a half-gap of the outside radius of the pole. At greater radii, the linear decrease is not maintained due to the influence of the outside edge of the pole. It is necessary to know what departures from linearity are permissible.

In order that the conditions necessary for vertical focusing of the beam may be maintained, the field must decrease with increasing radius. The exact amount depends on the height of the ion beam and other factors connected with the ion orbits.

The condition on the maximum rate at which the field may decrease is more satisfactory. The Berkeley Group* has shown that at the radius where $n = 1/5$, n being defined as

$$n = -\frac{r}{H} \frac{dH}{dr} \quad (9)$$

where r = radius

H = field

a vertical defocusing of the ion beam occurs. Thus the condition $n \ll 1/5$ must be satisfied for all radii less than the maximum which is to be used. The problem of shaping the shim edge and pole side is one of obtaining a pole which will give the required energy in the ion beam with the least steel, copper, etc. Any attempt at a rigorous solution to this problem must include a determination of the field in the steel. The usual approximations for treating the steel, i.e., assuming infinite permeability or assuming complete saturation, cannot be applied to this problem because the flux density in the corner of the pole varies from moderate to well over saturation. A simple approach to this problem, which has proved quite fruitful, is to determine the potential of a satisfactory magnetic field in a region free of singularities and then to decide whether the field could be produced by a practical steel pole.

This procedure is only valuable for the suggestions which the results offer. The design of the pole tip must be completed by experimental, cut and try procedures. The treatment of the problem in three dimensions was attempted. The equations for determining the field due to poles which are right circular cylinders resting on yokes, which are infinite planes, (with any distribution of potential on the pole sides and yokes) is given in Appendix II. A glance at this work will indicate that the labor involved is inconsistent with the usefulness of the results.

The general principles of the shim edge and pole side are the same in a two-dimensional as in a three-dimensional magnet. Thus a study of the problem of terminating the field of a two-dimensional magnet would offer valuable suggestions which could be used in constructing a cyclotron magnet.

There is a procedure which keeps the simplicity of a two-dimensional analysis and also gives numerical results which are approximately correct for a three-dimensional magnet. The field problem is solved in two dimensions and interpreted as the field present in a plane which contains the axis of the three-dimensional magnet. The errors made in this way depend on the geometry under consideration and the results required. For example, Fig. 1.5 shows portions of two flux maps for the same magnet. These maps were made with the standard procedure for mapping in air and copper. The top figure is a three-dimensional map, and the shaded area is the first tube of flux which enters the side of the pole. The lower figure is a two-dimensional map with the corresponding tube shaded. The points marked

*D. C. Sewell, L. Henrich, and J. Vale, Phys. Rev., 72: 739(1947); D. M. Dennison and T. H. Berlin, Phys. Rev., 70: 58(1946).

$y = 3$ are three half-gaps from the mid-plane, and at these points the difference in the leakage indicated by the two maps is about 0.16 tube. Each tube of the three-dimensional map carries about 1/6 of the total flux. Thus the error in the total flux at $y = 3$ made by interpreting the two-dimensional map as the field of a three-dimensional magnet is less than 3 per cent. Further approximations are introduced in the subsection, Square Equipotential.

It is necessary to bear in mind that the primary object of these calculations is to obtain reasonable goals for experimental design and to suggest methods for obtaining these goals.

Square Equipotential

The field due to a square equipotential is obtained with a Schwartz transformation. It is presented here because it is a simple example which illustrates the approximations and calculations which will be employed in the later problems. It is clear that a square steel corner will not produce the field derived below, but it is probable that a suitably shaped shim edge and pole side could produce a field very similar to this one.

The field $V = -\frac{\theta_1}{\pi}$ is analytic in the complex z_1 plane, $z_1 = r_1 e^{i\theta_1} = x_1 + iy_1$. By means of the transformation,

$$\frac{dz}{dz_1} = \frac{1}{\pi} \frac{\sqrt{z_1 + 1}}{z_1} \quad (10)$$

this field may be mapped in the complex z plane. This gives the field of an equipotential with a square corner. This process is shown in Fig. 1.6.

The integral of Eq. 10 is

$$z = \frac{1}{\pi} \left[2 \sqrt{z_1 + 1} + \log \left(\frac{\sqrt{z_1 + 1} - 1}{\sqrt{z_1 + 1} + 1} \right) \right] \quad (11)$$

The field in the z plane is

$$-\frac{\partial V}{\partial z} = \frac{1}{\pi} \left(\frac{\partial z}{\partial \theta_1} \right)^{-1} = \frac{-i}{\sqrt{z_1 + 1}} = H \quad (12)$$

and the field in the mid-plane, $z_1 = x_1 > 0$ is

$$H_y = -\frac{\partial V}{\partial y} = -\frac{\partial V}{\partial z} \frac{\partial z}{\partial y} = \frac{1}{\sqrt{x_1 + 1}} \quad (13)$$

Note that this is a homogeneous rather than a linearly decreasing field apart from the effect of the edge of the gap. The problem which will be studied in the following paragraph is that of terminating a homogeneous field in such a way that the field is monotone, decreasing as the edge of the gap is approached. This greatly simplifies the problem and does not reduce the usefulness of the results.

To determine the point where $n = \frac{1}{5}$ the derivative of the field with respect to x is required.

$$\frac{\partial H_y}{\partial x} \frac{\partial H_y}{\partial x_1} \frac{\partial x_1}{\partial x} = -\frac{\pi}{2} \frac{x_1}{(x_1 + 1)^2} \quad (14)$$

Recalling the definition of n , Eq. 9,

$$n = \frac{r}{H_y} \frac{dH_y}{dx} = \frac{\pi r}{2} \frac{x_1}{(x_1 + 1)^{3/2}}$$

r is the radius at which n is computed; if m is the radius to the edge of the pole, $x = 0$, measured in one half-gap, then

$$r = m + \frac{1}{\pi} \left[2 \sqrt{x_1 + 1} + \log \left(\frac{\sqrt{x_1 + 1} - 1}{\sqrt{x_1 + 1} + 1} \right) \right] \quad (15)$$

The equation for x_1 where $n = 0.2$ becomes

$$n = \frac{1}{5} = \frac{m\pi}{2} \frac{x_1}{(x_1 + 1)^{3/2}} + \frac{x_1}{x_1 + 1} + \frac{1}{2} \frac{x_1}{(x_1 + 1)^{3/2}} \log \left(\frac{\sqrt{x_1 + 1} - 1}{\sqrt{x_1 + 1} + 1} \right) \quad (16)$$

Cyclotrons are usually built with the ratio of pole diameter to gap height m , of about 9. By taking $m = 9$, the solution to Eq. 15 is

$$x_1 = 0.0165 + \quad (17)$$

and using this result in Eq. 11 gives

$$x = -1.108 \quad (18)$$

The field at this point is given by Eq. 13, and the radius is given by Eq. 15.

$$H_0 = 0.9917 \quad \rho = 7.89 \quad \frac{\rho}{m} = 0.877 \quad (19)$$

where ρ is the value of r where $n = 0.02$.

The leakage flux must also be computed. To obtain the leakage, write the imaginary part of Eq. 11 with

$$\begin{aligned} -z_1 &= u_1 \quad 1 < u_1 \\ iz = y &= \frac{2}{\pi} \sqrt{u_1 - 1} + \frac{1}{\pi} \arctan \frac{2\sqrt{u_1 - 1}}{u_1 - 2} \end{aligned} \quad (20)*$$

Equation 20 is solved for various values of u_1 , thus giving the point on the pole where the leakage into the side totals $(\log u_1)/\pi$ tubes per half-gap of pole circumference.

$\pi \times \text{leakage}$	u_1	y
0.5	$e^{0.5}$	1.082
1.0	e	1.249
1.5	$e^{1.5}$	1.503
2.0	e^2	1.85
2.5	$e^{2.5}$	2.31
3.0	e^3	2.92
3.5	$e^{3.5}$	3.71

A graph of these results is given in Fig. 1.7.

*Equation 20 may be expressed by

$$y = \frac{2\sqrt{u_1}}{\pi} \sum_{n=0}^{\infty} \frac{(2n)! u_1^{-n}}{2^{2n} (n!)^2 (2n-1)^2} \quad (21)$$

which is convenient for large u_1 .

By taking again the example $m = 9$, the average flux density in the pole, in a plane of constant y , can be computed. Since the accuracy of the calculation is limited when the two-dimensional analysis is applied to the three-dimensional problem, certain approximations will be made. By taking the field inside the useful radius as one, the useful flux is, from Eq. 19,

$$\pi(7.89)^2 = 196 \text{ tubes}$$

The leakage flux into the top of the pole is (Eq. 17) $(-\frac{1}{\pi} \log x_1)$ per half-gap of mean circumference, or

$$2\pi \frac{7.89 + 9}{2} \frac{1}{\pi} \log \frac{1}{0.0165} = 69 \text{ tubes}$$

and the leakage into the side of the pole from the corner to three half-gaps from the mid-plane is (see Fig. 1.7),

$$2\pi \times 9 \frac{3.05}{\pi} = 55 \text{ tubes}$$

The flux efficiency may be defined as the "useful" flux divided by the total flux at some plane in the base of the pole. In this work, the plane is taken as the $y = 3$ plane since the two-dimensional analysis is fairly accurate to this distance. Thus,

$$\text{flux efficiency} = F = \frac{\pi \rho^2 H_0}{\phi_{y=3}} = 0.61$$

where H_0 = field where $n = 1/5$

ρ = radius where $n = 1/5$

The average flux densities are, at $y = 1$, flux density = 1.04, and, at $y = 3$, flux density = 1.26 = B_B .

To summarize, a magnet with a square equipotential which has a ratio of pole diameter to gap height, m , of 9, has a useful radius of 87.7 per cent of the pole radius, a flux efficiency of 61 per cent, and a flux density of 1.26 per cent at a distance of three half-gaps from the mid-plane. The field at the point where $n = 0.2$ is $H_0 = 0.9917$.

It will be noted that the effect of the coils which energize the magnet has not been mentioned in the above discussion. Neglecting the coils is a reasonable approximation if most of the coil is below the point where the flux is determined. For example, the analytic two-dimensional treatment given above indicates that one tube of flux enters the side of the pole between the top of the pole, $y = 1$, and the point on the side where $y = 3$, as is indicated in Fig. 1.7. The two-dimensional flux map for a similar geometry except for the presence of a yoke and a coil (see Fig. 1.5) indicates that one tube enters between the top of the pole and the point $y = 2.9$. This difference is hardly significant in view of the approximate nature of the flux map.

When the effect of the coil cannot be neglected the two-dimensional analysis can still be used. An example of a pole and yoke energized by a coil which consists of a single wire is given in Appendix III.

Criterion for Comparing Pole Tips

Before other pole tips are studied, it is necessary to have a basis for comparing the results. A logical basis for such a comparison is the energy of the ion beam which can be obtained with a given pole base, yoke, coil, and power supply. A simple combination of numbers which approximately satisfies this condition is

$$E = \frac{\pi H_0^2 \rho^2}{\phi_B B_B} \quad (22)$$

where E = "efficiency" of pole tip

H_0 = field where $n = 0.2$

ρ = radius where $n = 0.2$

ϕ_B = maximum flux in the pole base

B_B = maximum flux density in the pole base

The value of the numerator determines the energy of the ion beam and the factors in the denominator determine the quantity of flux in the base of the pole and the size of the base of the pole. The maximum flux in the pole base determines the size of yoke which will carry the flux with a given ampere-turn loss. The flux density in the pole base determines the ampere-turn loss in this region; the size of the pole base determines the size of the coil. The only factors which enter into the cost of the cyclotron magnet that are not taken into account by Eq. 22 are variations in the ampere-turn losses in the pole tips and gap.

Eq. 22 may be conveniently written

$$E = FD \quad (23)$$

where F = flux efficiency

$$= \frac{\text{"useful flux"}}{\text{total flux}}$$

$$= \frac{\pi \rho^2 H_0}{\phi_B}$$

D = relative density

$$= \frac{\text{"density in the gap"}}{\text{flux density in the base}}$$

$$= \frac{H_0}{B_B}$$

The upper limit to the efficiency, E , is achieved by a thin coil which is wound over the gap. With this arrangement all of the flux is used, and the maximum density in the pole is the same as the density in the gap. Thus the upper limit for E is one and is obtained only with an arrangement that is useless in a conventional cyclotron.

The derivations that follow are aimed primarily at obtaining E for the fields that are considered. First, the important role of D is illustrated. Arrangements are discussed which give a low F but a high D . This is done mostly as a further introduction to the mathematical methods employed. The second case considered is a pole which will give a useful field shape over a wide range of fields. The E for this pole is taken as a lower limit for a good design. Under many circumstances, the ability to operate at any field strength should be sacrificed in order to obtain higher energy. To obtain an upper limit for the pole tip efficiency, E , a shim edge which crosses the gap is considered. Following this is a derivation which gives the maximum useful fraction of the radius which can be obtained with the conventional raised-edge shim. The final derivation suggests a method by which the useful fraction of the radius and the pole-tip efficiency may be made greater.

As a matter of convenience, in the derivations ϕ_B and B_B will always be determined at the plane which is three half-gaps from the mid-plane; however, for work on actual magnets, the maximum flux and flux density in the pole base are more appropriate quantities. It is assumed that the $V = 1$ equipotential coincides at $y = 3$ with the steel surface which would produce the field.

Equipotentials Mathematically Related to Square Equipotential

One of the equipotentials of the transformation given in Eq. 10 is the square equipotential. However, it is clear that there are a whole family of equipotentials corresponding to values of θ_1 less than π , which may be studied. The fields associated with these equipotentials are not good, that is, the flux efficiency is low; but, since the ratio D of the density in the gap to the density in the base is greatly increased, the pole-tip efficiency, E , is improved. In this paragraph, this family of pole tips is considered.

The transformation is the same as that used in the subsection, Square Equipotential, except for a constant which adjusts the half-gap to one,

$$z = \frac{1}{\pi(1-2\delta)} \left[2 \sqrt{z_1 + 1} + \log \left(\frac{\sqrt{z_1 + 1} - 1}{\sqrt{z_1 + 1} + 1} \right) \right] \quad (24)$$

where $\pi\delta$ = the angle between the side of the pole and the y axis as is indicated in Fig. 1.8.

The equation for the field at $y = 0$ is

$$H = \frac{1}{\sqrt{x_1 + 1}} \quad (25)$$

The equation for n is

$$n = \frac{\pi(1-2\delta)}{2} r \frac{x_1}{(x_1 + 1)^{3/2}} \quad (26)$$

The point to be called the edge of the pole is ambiguous and is arbitrarily taken at $x = 0$ for simplicity. This makes the equation for n

$$\frac{1}{5} = n = \frac{\pi(1-2\delta)}{2} m \frac{x_1}{(x_1 + 1)^{3/2}} + \frac{x_1}{x_1 + 1} + \frac{1}{2} \frac{x_1}{(x_1 + 1)^{3/2}} \log \left(\frac{\sqrt{x_1 + 1} - 1}{\sqrt{x_1 + 1} + 1} \right) \quad (27)$$

The solution of this equation for $x_1 \big|_{n=0.2}$ is given in Fig. 1.9. The values for ρ and ρ/m are found by Eq. 24 and given in Fig. 1.9.

In order to determine the leakage at $y = 3$ a series expansion of the imaginary part of Eq. 24 with $\theta_1 = 2\pi\delta$ is convenient.

$$3 = y = \frac{2 \sqrt{r_1}}{\pi(1-2\delta)} \sum_{n=0}^{\infty} \frac{(2n)! r_1^{-n} \sin \pi \left[\frac{1}{2} + (2n-1)\delta \right]}{2^{2n} (n!)^2 (2n-1)^2} \quad (28)$$

Good accuracy is obtained by solving this equation for r_1 with only two terms

$$r_1 \big|_y = \left(\frac{b + \sqrt{b^2 - 8}}{4} \right)^2 \quad (29)$$

$$b = \frac{\pi y (1-2\delta)}{\sin \frac{\pi}{2} (1-2\delta)}$$

The flux leakage into the pole between $r_1 = 1$ and $r_1 = r_1 \big|_y$ per average half-gap of circumference is

$$\phi'_1 = \frac{1}{\pi(1-2\delta)} \log r_1 \big|_y$$

The radius of the pole at $y = 3$ is obtained by substituting the value of $r_1 \big|_{y=3}$ obtained above into the equation

$$r_3 = m + \frac{2 \sqrt{r_1}}{\pi(1-2\delta)} \sum_{n=0}^{\infty} \frac{(2n)! r_1^{-n} \cos \pi \left[\frac{1}{2} + (2n-1)\delta \right]}{2^{2n} (n!)^2 (2n-1)^2} \quad (30)$$

the series being the real part of Eq. 24. The leakage into the side of the pole is found by multiplying Eq. 30 by the average circumference which is approximately $\pi(r_3 + m)$.

$$\phi_1 = \left. \frac{r_3 + m}{1 - 2\delta} \log r_1 \right|_3 \quad (31)$$

The leakage into the face of the pole is found by multiplying $-\frac{1}{\pi(1-2\delta)} \log x_1 \Big|_{n=0.2}$ by the average circumference which is approximately $\pi(m + \rho)$.

$$\phi_2 = \left. \frac{m + \rho}{1 - 2\delta} \log x_1 \right|_{n=0.2} \quad (32)$$

The remaining flux in the region $r = 0$ to $r = \rho$ is

$$\phi_3 = \pi \rho^2 \quad (33)$$

The flux efficiency is determined from Eqs. 31, 32, and 33 by

$$F = \frac{\pi \rho^2 H_0}{\phi_1 + \phi_2 + \phi_3} \quad (34)$$

The values of F as a function of δ are given in Fig. 1.10 for $m = 9$. The flux density at $y = 3$ is given by

$$B_B = \frac{\phi_1 + \phi_2 + \phi_3}{\pi r_3^2} \quad (35)$$

The solution of this equation for $m = 9$ is given in Fig. 1.11 along with the values of D . The values of E found by Eq. 22 are given in Fig. 1.12 along with the values of H_0 . There is a real improvement in the energy obtainable with a given amount of equipment by using a pole-tip shape with δ greater than zero. For example, if δ is 0.1, then the angle which the side of the pole makes with the vertical is 18 deg, the field at the point where $n = 0.2$ is 98.94 per cent of the uniform field in the gap, the useful radius is 85.8 per cent, and the flux efficiency is 55.8 per cent. Because the relative flux density, $D = H_0/B_B$, is higher than with a cylindrical pole, 0.93 instead of 0.79, this pole can be operated with a higher field in the gap without saturating the steel, and a 9 per cent higher energy in the cyclotron beam can be obtained than with a cylindrical pole.

Rounded Equipotentials

Only a very restricted class of equipotential shapes was treated in the previous paragraph. A much more general set is developed by the following transformation:

$$z = 2C \sqrt{z_1 + 1 + a} + 2C\lambda \sqrt{z_1 + 1 - a} + C \sqrt{1 + a} \log \left(\frac{\sqrt{z_1 + 1 + a} - \sqrt{1 + a}}{\sqrt{z_1 + 1 + a} + \sqrt{1 + a}} \right) + C\lambda \sqrt{1 - a} \log \left(\frac{\sqrt{z_1 + 1 - a} - \sqrt{1 - a}}{\sqrt{z_1 + 1 - a} + \sqrt{1 - a}} \right) \quad (36)$$

where

$$C = \frac{1}{\pi(\sqrt{1 + a} + \lambda \sqrt{1 - a})}$$

A typical shape obtained is sketched in Fig. 1.13. There are a double infinity of shapes depending on a and λ . This means that the end points of the rounded corner

$$z = (x_\alpha + i) \quad z = iy_\alpha$$

may be put in any desired place in the z plane. By letting

$$z_1 = -(1 - a)$$

and

$$z_1 = -(1 + a)$$

in Eq. 36, the equations for x_α and y_α are obtained.

$$x_\alpha = 2C\sqrt{2a} - C\sqrt{1+a} \log \left(\frac{\sqrt{1+a} + \sqrt{2a}}{\sqrt{1+a} - \sqrt{2a}} \right) \quad (37)$$

$$y_\alpha = 2C\lambda\sqrt{2a} + C\pi\sqrt{1+a} + C\lambda \arctan \left(\frac{2\sqrt{2a(a-a)}}{3a-1} \right) \quad (38)$$

The potential, V , is set equal to $-\theta_1/\pi$ as in the subsection, Square Equipotential. The field in the z plane at $y = 0$ is

$$H = \frac{1}{\pi C} \frac{1}{\sqrt{x_1 + 1 + a} + \lambda \sqrt{x_1 + 1 - a}} \quad (39)$$

The equation for n is

$$n = -\frac{r}{H} \frac{dH}{dx} = x_1 \frac{r}{2C} \left(\frac{1}{\sqrt{x_1 + 1 + a} + \lambda \sqrt{x_1 + 1 - a}} + \frac{\lambda}{\sqrt{x_1 + 1 + a} - \lambda \sqrt{x_1 + 1 - a}} \right)$$

If the nominal radius of the pole, m , is the distance from the center of the pole to the point where the pole starts to curve, then

$$r = m - x_\alpha + x \quad (40)$$

The equation for n becomes

$$n = x_1 \left(\frac{1}{\sqrt{x_1 + 1 + a} + \lambda \sqrt{x_1 + 1 - a}} + \frac{\lambda}{\sqrt{x_1 + 1 + a} - \lambda \sqrt{x_1 + 1 - a}} \right) \left[1 + \frac{\left(\frac{m - x_\alpha}{c} + \sqrt{1+a} \log \frac{\sqrt{x_1 + 1 + a} - \sqrt{1+a}}{\sqrt{x_1 + 1 + a} + \sqrt{1+a}} \right)}{2(\sqrt{x_1 + 1 + a} + \lambda \sqrt{x_1 + 1 - a})} \right. \\ \left. + \frac{\lambda \sqrt{1-a} \log \left(\frac{\sqrt{x_1 + 1 - a} - \sqrt{1-a}}{\sqrt{x_1 + 1 - a} + \sqrt{1-a}} \right)}{2(\sqrt{x_1 + 1 + a} + \lambda \sqrt{x_1 + 1 - a})} \right] \quad (41)$$

This equation may be used to determine $x_1]_{n=1/5}$ conveniently by solving it for the factor x_1 which appears alone on the right-hand side. A guess at the value of x_1 in the remaining terms yields a much improved value, and this process may be repeated until the desired accuracy is obtained. The value of the radius where n is $1/5$ may be obtained from Eqs. 36 and 37. The leakage per average half-gap of circumference of the pole face is given by

$$\phi'_1 = \frac{1}{\pi} \log \left(\frac{1-a}{x_1]_{n=1/5}} \right) \quad (42)$$

and the leakage per average half-gap of circumference into the side of the pole to the point where $y = y_\alpha$ is given by

$$\phi'_2 = \frac{1}{\pi} \log \left(\frac{1+a}{1-a} \right) \quad (43)$$

From these quantities it is a straightforward calculation to determine the performance of the pole.

As an example, the set of pole tips with $y_\alpha = 3$ and $m = 9$ was studied. Equation 38 was solved for λ as a function of a with $y_\alpha = 3$. The values of x_α were found by Eq. 37. These results are shown in Figs. 1.14 and 1.15, respectively. Equation 41 was then solved for $x_1]_{n=0.2}$. This result, shown in

Fig. 1.15 was used to find ρ , which is plotted in Fig. 1.16 as a function of x_α . The leakage was calculated from the formulas

$$\left. \begin{aligned} \phi_1 &= \pi(m + \rho) \frac{1}{\pi} \log \left(\frac{1-a}{x_1]_{n=0.2}} \right) \\ \phi_2 &= \pi(2m + x_\alpha) \frac{1}{\pi} \log \left(\frac{1+a}{1-a} \right) \end{aligned} \right\} \quad (44)$$

which follow from Eqs. 42 and 43. (Using the straight average for the circumference tends to make the leakage somewhat too large.) The flux efficiency was determined by

$$F = \frac{H_0 \pi \rho^2}{\pi \rho^2 + \phi_1 + \phi_2} \quad (45)$$

and the flux density at $y = 3$ by

$$B_B = \frac{\pi \rho^2 + \phi_1 + \phi_2}{\pi(m + x_\alpha)^2} \quad (46)$$

and the relative energy obtained by

$$E = \frac{H_0 F}{B_B} \quad (47)$$

These results are shown in Fig. 1.17.

It should not be concluded from Fig. 1.17 that 0.524 is the maximum value of E which can be obtained with the rounded pole tip, since higher values can be obtained with larger y_α .

The pole with $y_\alpha = 3$ and $x_\alpha = -1.8$ is of special interest because it represents a pole which can be built and would have a field shape independent of the field strength (to 18,000 gauss). This is due to the fact that the flux density at the surface of the steel is not excessive at any point. It can be shown from the equation for the field, Eq. 39, that the maximum flux density at the steel surface is

$$\frac{\sqrt{1+a} + \lambda \sqrt{1-a}}{\sqrt{2a}}$$

which turns out to be about 1.12 times the gap density for this pole. The average density in the pole at $y = 3$ is 1.07 which is again reasonably low. The pole has a useful radius of 92.5 per cent and an E of 0.524.

Shim That Crosses the Gap

The ability to operate over a wide range of fields, which was possessed by the pole just discussed, is usually not required in a cyclotron magnet and should be sacrificed to obtain a higher pole-tip efficiency.

In this paragraph an upper limit for E, the pole-tip efficiency, is determined by considering a shim which crossed the gap at the outside circumference of the pole.

A shim which crosses the gap at the edge of the pole can have its reluctance so adjusted that the field is useful to the full radius of the gap. This amounts to putting a potential divider across the edge of the gap which keeps the field homogeneous inside the gap. Although this arrangement would obstruct the gap at the edge, it furnishes an interesting upper limit to the pole-tip efficiency.

The field inside the gap is homogeneous; it is therefore only necessary to treat the field outside the gap. One method for studying this field is to place the edge of one pole in the complex z_1 plane from $x_1 = -\infty$ to $x_1 = -1$ and the edge of the other pole from $x_1 = 1$ to $x_1 = \infty$. The gap is then between $x_1 = -1$ and $x_1 = +1$ and the field to be investigated in the upper z_1 half-plane. The potential distribution on the x_1 axis is

$$\left. \begin{aligned} V_0 &= +1 & -\infty < x_1 < -1 \\ V_0 &= -x_1 & -1 < x_1 < 1 \\ V_0 &= -1 & 1 < x_1 < \infty \end{aligned} \right\} \quad (48)$$

The real axis is folded at -1 and $+1$ through an angle $\pi/2$ by a Schwartz transformation. The transformation is

$$\left. \begin{aligned} dz &= \frac{2dz_1}{\pi \sqrt{z_1^2 - 1}} \\ z &= -i + \frac{2}{\pi} \log \left(z_1 + \sqrt{z_1^2 - 1} \right) \end{aligned} \right\} \quad (49)$$

which integrates to

This process is illustrated in Fig. 1.18. Due to the symmetry, only the region from $0 \leq y \leq 1$ need be considered.

The potential at $x = 0$ can be found by substituting the second of the expressions in Eq. 48 into the imaginary part of Eq. 49,

$$\left. \frac{\pi}{2} (1 + y) \right]_{x=0} = \arctan \frac{\sqrt{1-x_1^2}}{x_1} = \arctan \frac{\sqrt{1-V_0^2}}{-V_0} \quad (50)$$

which can be solved for V_0

$$V_0]_{x=0} = \sin \frac{\pi y}{2} \quad (51)$$

The potential in the z plane can be expressed in the form

$$\left. \begin{aligned} V_0 &= y - \sum_{m=1}^{\infty} A_m \sin m\pi y e^{-m\pi x} \\ \text{In order to satisfy Eq. 51, the } A_m \text{ are} \\ A_m &= \frac{2(-1)^m}{\pi m(4m^2 - 1)} \end{aligned} \right\} \quad (52)$$

The point where $n = 0.2$ need not be found since the useful radius is the full radius of the pole, m . The leakage flux is most conveniently determined in the z plane by integrating the flux crossing the boundary

$$\begin{aligned} y &= 0 & 0 \leq x \leq M & \quad M \rightarrow \infty \\ y &= 1 & x_3 \leq x \leq M \end{aligned}$$

where x_3 is the value of x when $x_1 = 3$ as determined by Eq. 49.

This treatment can be readily generalized to include any of the equipotentials in the z_1 plane as possible pole shapes, with a square pole corner included as a special case. The angle $\pi\delta$ is the angle between the x_1 axis and the side of the pole. Equation 48 becomes

$$\begin{aligned} a &= \frac{1}{1 - 2\delta} & \begin{aligned} V &= a & -\infty < x_1 < -a \\ V &= -ax_1 & -a < x_1 < a \\ V &= -a & a < x_1 < \infty \end{aligned} \end{aligned} \quad (53)$$

Equation 49 may be replaced by

$$z = -i + \frac{2}{\pi} \log \frac{z_1}{a} + \sqrt{\frac{z_1^2}{a^2} - 1} \quad (54)$$

Equations 51 and 52 become

$$V = a \sin \frac{\pi y}{2} \quad (55)$$

$$\left. \begin{aligned} V &= ay - a \sum_{m=1}^{\infty} A_m \sin m\pi y e^{-m\pi x} \\ A_m &= \frac{2(-1)^m}{\pi m(4m^2 - 1)} \end{aligned} \right\} \quad (56)$$

The leakage may be determined by integrating the flux in the z plane:

$$\phi = \int_0^b a \, dx + \int_0^\infty \left. \frac{\partial(V - ay)}{\partial y} \right]_{V=0} dx - \int_b^\infty \left. \frac{\partial(V - ay)}{\partial y} \right]_{V=1} ds \quad (57)$$

where

$$b = x \Big|_{x_1=3}^{V=1}$$

The signs are chosen so as to make ϕ' positive. To solve for b note that approximately

$$y_1 = x_1 \tan \pi \delta \quad (58)$$

Equation 54 may be written

$$\cosh \frac{\pi}{2} (z + i) = \frac{z_1}{a} \quad (59)$$

Substituting $z = b + iy$ and Eq. 58 into Eq. 59 gives two equations from which y can be eliminated to give

$$\frac{x_1^2}{a^2} \left(\tanh^2 \frac{\pi b}{2} + \tan^2 \pi \delta \right) = \sinh^2 \frac{\pi b}{2} \quad (60)$$

The solution of Eq. 60 for b as a function of δ is given in Fig. 1.19. Integrating Eq. 57 gives

$$\phi' = ab - a \sum_{m=1}^{\infty} A_m + a \sum_{m=1}^{\infty} A_m \cos m\pi(1 - 2\delta) e^{-m\pi b} \quad (61)$$

The last integral has been evaluated approximately but its contribution is so small that it will be neglected. The sum of the series $-\sum_{m=1}^{\infty} A_m$ is $2/\pi \cdot 0.3068$; thus the leakage per average half-gap of circumference is

$$\phi' = a(b + \frac{2}{\pi} 0.3068)$$

The total leakage at $x_1 = 3$ is

$$\phi_3 = (2m + 3 \tan \pi \delta) a(\pi b + 2 \cdot 0.3068) \quad (62)$$

The flux efficiency is

$$F = \frac{\pi m^2}{\pi m^2 + \phi_3} \quad (63)$$

and the flux density at $x_1 = 3$ is

$$B_B = \frac{\pi m^2 + \phi_3}{\pi(m + 3 \tan \pi \delta)^2} \quad (64)$$

F and B_B are plotted as functions of δ for $m = 9$ in Fig. 1.19. E , the pole-tip efficiency, is computed by Eq. 22 and is plotted as a function of δ in Fig. 1.20.

It is clear in the case of a "perfect" shim, as was true with no shim (see subsection, Equipotentials Mathematically Related to Square Equipotential), that more energy can be obtained with a given amount of material and power if a tapered pole is used at high field than if a cylindrical pole is used at a lower field. It is also apparent that a shim is valuable. For example with $\delta = 0.15$ the angle of the side of the pole with the vertical is 27 deg, the useful radius and field are 100 per cent, the flux efficiency is 0.71, and the flux density three half-gaps from the mid-plane is 1.02 times the homogeneous field in the gap. The energy which can be obtained with this arrangement is 145 per cent of that which can be obtained with a square-equipotential corner.

Pole with Raised Edge

The simplicity of the treatment justifies considering a pole with a raised edge to determine the fraction of the radius which can be obtained with this type of shim. This equipotential is generated by the transformation

$$z = \frac{1}{\pi} \left[\frac{2 \sqrt{z_1 + 1}}{a + 1} + \log \left(\frac{\sqrt{z_1 + 1} - 1}{\sqrt{z_1 + 1} + 1} \right) \right] \quad (65)$$

This reduces to Eq. 11 when $a = 0$. The equipotential produced is shown in Fig. 1.21.

It can be shown that the field on the x axis is monotone, decreasing with x , provided that

$$a \leq 1 \quad (66)$$

The transformation which is considered, therefore, is

$$z = \frac{1}{\pi} \left[\sqrt{z_1 + 1} + \log \left(\frac{\sqrt{z_1 + 1} - 1}{\sqrt{z_1 + 1} + 1} \right) \right] \quad (67)$$

since the useful radius is extended as far as possible by making $a = 1$. The field is given by

$$H = \frac{2 \sqrt{x_1 + 1}}{x_1 + 2} \quad (68)$$

The derivative of H with respect to x is

$$\frac{dH}{dx} = - \frac{2\pi x_1^2}{(x_1 + 2)^3} \quad (69)$$

n is given by

$$n = - \frac{r}{H} \frac{dH}{dx} = \frac{\pi x_1^2 r}{(x_1 + 2)^2 \sqrt{x_1 + 1}} \quad (70)$$

Since r is

$$r = m + \frac{1}{\pi} \left[\sqrt{x_1 + 1} + \log \left(\frac{\sqrt{x_1 + 1} - 1}{\sqrt{x_1 + 1} + 1} \right) \right] \quad (71)$$

the equation for n becomes

$$n = \frac{x_1^2}{(x_1 + 2)^2} \left[1 + \frac{1}{\sqrt{x_1 + 1}} \left[\pi m + \log \left(\frac{\sqrt{x_1 + 1} - 1}{\sqrt{x_1 + 1} + 1} \right) \right] \right] \quad (72)$$

With $n = 1/5$ and $m = 9$ this equation gives $x_1 = 0.200$. The other interesting quantities determined in the usual way are given below:

$$\begin{array}{ll} \rho = 8.365 & \rho/m = 0.919 \\ H_0 = 0.996 & F = 0.667 \\ B_B = 1.291 & E = 0.511 \end{array}$$

This method of increasing the useful radius is valuable as is indicated by the value of E . The height of the edge in the z plane is 0.182 half-gap, i.e.,

$$y = \frac{1}{\pi} + \frac{1}{2}$$

at the tip of the raised edge. It does not appear worth while to extend these calculations to the related family of equipotentials.

Poles Obtained by Superposition

The variety of pole-tip shapes which can be obtained with Schwartz transformations which integrate into simple functions is limited. Since it is desirable to have a better estimate for the expected pole-tip efficiency, E , a few pole-tip shapes obtained by adding two or more simple fields will be considered.

The basic field will be taken as that due to a square-equipotential corner which is given in terms of the parameter x_1 in the subsection, Square Equipotential,

$$H_s = \frac{1}{\sqrt{x_1 + 1}}$$

$$x = \frac{1}{\pi} \left[2\sqrt{x_1 + 1} + \log \left(\frac{\sqrt{x_1 + 1} - 1}{\sqrt{x_1 + 1} + 1} \right) \right] \quad (73)$$

The first field which will be added to that of the square equipotential corner is a dipole at $X = 0$, $Y = 1$ which is oriented so that its axis is parallel to the y axis. The potential of such a dipole may be derived from the potential of a line charge and its image in the mid-plane, by differentiating with respect to Y . The potential of a line charge and its image is

$$V = k \log \frac{(x + X)^2 + (y + Y)^2}{(x + X)^2 + (y - Y)^2} \quad (74)$$

where x, y = coordinates in the z plane

X, Y = coordinates of line charge in the z plane

Differentiating gives

$$V_2 = \frac{ky(x^2 + y^2 - Y^2)}{x^4 + 2x^2(y^2 + Y^2) + (y^2 - Y^2)^2} \quad (75)$$

and the field in the mid-plane is

$$H_2 = - \left. \frac{\partial V}{\partial y} \right|_{y=0} = \frac{k(Y^2 - x^2)}{(x^2 + Y^2)^2} \quad (76)$$

In order to find the value of k , the derivatives of H_s and H_2 with respect to x were determined, and the maximum value of k which made the derivative of $H_s + H_2$ less than or equal to zero was selected. This process, which assures a monotone decreasing field with increasing x , is shown graphically in Fig. 1.22. The value of k obtained is 0.0694.

In order to obtain a family of pole-tip shapes the parameter δ is introduced. $\pi\delta$ is the angle between the side of the pole and the vertical as in the subsection, Equipotentials Mathematically Related to Square Equipotential. The change of scale required to adjust the half-gap to one is taken care of by the change of variable

$$x_\delta = \frac{x}{1 - 2\delta} \quad (77)$$

In this notation the formula for n is

$$n = \frac{r}{H} \frac{dH}{dx_\delta} \quad r = m + x_\delta \quad (78)$$

This was solved numerically for x_δ as a function of δ with $n = 0.2$. The results of this calculation are given in Fig. 1.23 as are the related quantities ρ and ρ/m . The quantity m is taken to be nine, as before.

The flux in the pole base and radius of the pole base are calculated as previously, except that the flux is slightly increased due to the presence of the dipole at the corner. The additional flux per half-gap of circumference may be determined by integrating H_2 with respect to x .

$$\int H_2 dx = \frac{kx}{1+x^2} \quad (79)$$

This integral vanishes if the limits are made infinite. Since the contribution of this term to the total flux is small, the maximum value of the integral was taken; that is

$$\int_{-1}^{+1} H_2 dx = k \quad (80)$$

The additional flux after the variable of integration is changed to x_δ is

$$\phi = \frac{2\pi mk}{1-2\delta} \quad (81)$$

The remaining information required was obtained in the subsection, Equipotentials Mathematically Related to Square Equipotential. The quantity B_B , the density in the base, at $y = 3$, is shown as a function of δ in Fig. 1.23. The quantities F , the flux efficiency, and E , the pole efficiency, are shown as functions of δ in Fig. 1.24.

The remaining problem in this calculation is to determine the shape of the equipotentials. Since the potential functions are known, this is straightforward. It is convenient to make the substitution

$$\sqrt{z_1 + 1} = u + iv \quad (82)$$

in Eq. 73. The equation is then divided into the real and imaginary parts,

$$x = 2u + \frac{1}{2} \log \frac{(u^2 + v^2 - 1)^2 + 4v^2}{[(u+1)^2 + v^2]^2} \quad (83)$$

$$y = 2v + \arctan \frac{2v}{u^2 + v^2 - 1} \quad (84)$$

If u and x or v and y are given, the solution of these equations is straightforward. The task of plotting the entire field was not undertaken, but the height of the shim at the edge of the pole was calculated by setting $x = 0$, assuming values for u in Eq. 83, and solving the resulting biquadratics for v by Table 1.1, Appendix I. Equation 84 was then solved for y and Eq. 82 for θ_1 , and the potential was obtained by

$$V = -\frac{\theta_1}{\pi(1-2\delta)} \quad (85)$$

The height of the $V = 1$ point at the edge of the pole is given as a function of δ in Fig. 1.24.

Although the flux density at this point has not been calculated, it is sufficiently high, as that the steel surface would have to be above the $V = 1$ surface due to the finite permeability of steel.

A further improvement in the useful radius can be made by putting a line quadrupole with its axis parallel to the y axis at $x = 0, y = 1$. This field does not keep n from becoming 0.2 at a smaller radius and must have a correction field added to prevent this.

The potential for the quadrupole can be determined by differentiating Eq. 75 with respect to Y ,

$$V_3 = \frac{kYy [(x^2 + Y^2)(3x^2 + 2y^2 - Y^2) - y^4]}{[x^4 + 2x^2(y^2 + Y^2) + (y^2 - Y^2)^2]^2}$$

The field on the mid-plane, $y = 0$, is

$$H_3 = - \left. \frac{\partial V}{\partial y} \right|_{y=0} = \frac{kY(Y^2 - 3x^2)}{(x^2 + Y^2)^3}$$

The derivative of H_3 with respect to x was used to find the value of k which makes H a monotone decreasing function of x . As noted above, it was found necessary to decrease the derivative in the region $-2 < x < -1$ by adding a dipole at $x = -0.8, y = 1$ of strength $k = 0.0218$. The strength of the quadrupole used was $k = 0.090$. The derivative of the resulting field with respect to x as a function of x is given in Fig. 1.25. For this discussion, it does not appear worth while to make a better adjustment of the constants.

The value of x_δ where $n = 0.2$ was determined numerically, as before. The field at $x_\delta]_{n=0.2}$ remains essentially constant at $H_0 = 0.977$. The flux in the pole base was assumed to be the same as in the previous derivation. The flux efficiency, F , and the pole-tip efficiency, E , were calculated on this basis. The above results and also the related results, ρ and ρ/m , are shown in Fig. 1.26. The useful radius, ρ , is given by $m + x_\delta]_{n=0.2}$ and the nominal radius m was taken as nine.

The net flux which crosses the gap is not increased by dipoles, quadrupoles, etc. However, the field developed by them may be of such a nature that it cannot be produced by a pole tip made of steel. This could be judged if a map of the potential field were made. In a more elaborate analysis, the potential field could be given a structure which could certainly be produced by a steel pole. This could be done, for example, by adding appropriate line charges. A line charge and its image in the mid-plane are associated with flux crossing the gap; therefore, the leakage might increase. Thus the value of E obtained in this analysis may be somewhat high; also, B_B may be low and F high.

The distance from the $V = 1$ equipotential to the mid-plane is shown in Fig. 1.27 as is the approximate flux density at this point. It is not unreasonable to consider using a density of about two in the shim edge, but the shim will extend about twice as far toward the mid-plane as the $V = 1$ equipotential.

The distance of the shim from the mid-plane at $x = -0.4$ or $x_\delta - 0.4/1 - 2\delta$ was computed, assuming that the field due to the square-equipotential corner is homogeneous in this region and neglecting the contribution due to the dipole which is small in this region. These results are shown in Fig. 1.27. It is clear that just inside the raised edge, the gap is greater than the nominal gap.

Conclusions

The pole tip gives the field a satisfactory dependence on the position in the mid-plane of the gap and determines to a large extent the over-all performance of the magnet.

The potential associated with a field which decreases linearly with increasing radius, was given and a method for terminating this field in the center of the magnet was discussed.

A measure of pole-tip efficiency

$$E = \frac{\pi \rho^2 H_0}{\phi_B B_B} \quad (86)$$

gives interesting results. The maximum possible value for E is one. An unobtainably high value for magnets that have the coils well away from the gap is 0.71, and a value which can be reached with a pole which will work at "any" field is 0.52. An effort to obtain a value of the pole-tip efficiency, E , which could be used as a goal for experimental magnet design gave $E = 0.64$.

It is found that the useful radius with the conventional type of shim, that is, one with a raised outside edge, could be as high as 92 per cent of the pole radius. It is shown that this can be increased to about 96 per cent by cutting a groove into the pole face just inside of the raised edge. This result may be obtained in several ways and is unquestionably correct. In every case considered, a tapered pole was found to be very superior to a straight pole since a higher density in the gap may be used.

These data offer a firm foundation for the experimental studies which complete the design of the pole tip.

1.2 MAGNET COILS

Introduction

The coils of a cyclotron magnet supply the magnetomotive force necessary to operate the magnet. From the point of view of the one who uses a cyclotron, the coils are a necessary evil because they make the gap less accessible, and occupy space which might otherwise be used for apparatus.

In addition to affecting the convenience of the magnet, the size of the coils also affects the cost of the magnet. The economic factors are determined by the following:

1. The resistance of the coil is proportional to its mean circumference.
 2. An amount of steel approximately proportional to two times the height of one coil, plus the radial width of the coils, is required to complete the magnetic circuit.
 3. The reluctance of the pole and yoke depends in a similar way on the dimensions of the coils.
- For these reasons, the coil should be as small as is consistent with a reasonable power requirement. This goal is achieved by building the coils with a high average conductivity.

The average conductivity depends on the material used for the conductors, its temperature, and the fraction of the available volume occupied by the conducting material.

Under most circumstances, copper is the logical choice for the conducting material, because silver is too costly and other materials have lower conductivity.

The temperature of the conductors should be low in order to obtain good conductivity, and to keep heat out of the cyclotron room. The temperature of the conductors depends on the manner in which the heat is removed. The best cooling is achieved when the coolant is in contact with the conductor. The pressure required to move the coolant is a rapid function of the dimensions of the channels in which the coolant is flowing. In order to make the channels reasonably large without using much space in the coils, it is necessary that they be few in number. Direct cooling of the conductors with a few channels is accomplished by using large conductors with high currents.

The average conductivity also depends on the fraction of the coil volume occupied by the conductor; this fraction is known as the "space factor." The space factor is influenced by the type of cooling employed, as mentioned above, and by the space required for insulation. In a high-current low-voltage coil, insulation is required only to maintain mechanical separation of the conductors.

Two winding arrangements are considered which meet the requirements of a good coil as outlined above.

Winding Arrangements

The winding arrangements selected will depend largely on whether the machinery necessary for the conventional type of winding is available.

If a sufficiently large and powerful winding machine is available, the coil may be constructed from copper bars which are as high as the coil, and whose thickness depends on the width of the coil and the number of turns. Such a coil may be cooled with water tubes fastened to the top and/or bottom of the conductors. With adequate cooling at the edge, the temperature rise may be confined essentially to the rise within the conductor. The arrangements and equations for the temperature rise are given in

Fig. 1.28. This coil may also be cooled by oil flowing vertically between the turns. The problem of oil cooling is considered at length in later paragraphs.

A second type of coil which is more attractive, when the coil must be wound at the cyclotron site, is a flat-wound helix. In this case, the pieces of conductor are fabricated as sectors of an annulus and assembled in a continuous helix. This coil may be cooled by water tubes fastened to the inside and/or outside circumference with the conductors. In Fig. 1.29 the arrangements are given.

The equations for the temperature rises are, for the arrangement cooled with water tubes inside and outside the conductors,

$$\left. \begin{aligned} T - T_0 &= A \left[\frac{\log^2 \sqrt{\frac{r_0}{r_1}} - \log^2 \frac{r}{\sqrt{r_1 r_0}}}{\log^2 \frac{r_1}{r_0}} \right] & T_m - T_0 &= \frac{A}{4} \end{aligned} \right\}$$

for the arrangement cooled with water tubes inside,

$$\left. \begin{aligned} T - T_0 &= A \left[1 - \frac{\log^2 (r_1/r)}{\log^2 (r_1/r_0)} \right] & T_m - T_0 &= A \end{aligned} \right\} A = \frac{i^2 \rho}{8f^2 h^2 k}$$

and, for the arrangement cooled with water tubes outside,

$$\left. \begin{aligned} T - T_0 &= A \left[1 - \frac{\log^2 (r_0/r)}{\log^2 (r_1/r_0)} \right] & T_m - T_0 &= A \end{aligned} \right\}$$

This coil may also be cooled by oil flowing radially between the conductors.

Oil Cooling

Where oil cooling is used in the designs considered above, the oil flows in thin wide channels with the conductor composing at least one wall of the channel. In large coils of this type, the flow will probably turn out to be laminar.

It is a simple matter to determine the average temperature rise of the coolant, but it is also necessary to know how much the temperature of the cooled surface exceeds the average temperature or to be assured that this is small. An approximate equation for the surface rise which is satisfactory for most problems is derived below.

Assume that a fluid is flowing over a plate delivering constant heat per unit area and that the coolant enters at zero temperature and has a linear-velocity gradient with zero velocity at the surface delivering heat and a velocity v_0 at 1 cm from this surface. (See Fig. 1.30.)

Assume also that no heat is conducted in the y direction since, in any practical case, the temperature gradient in this direction is small compared to the gradient in the x direction. The diffusion equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{\sigma}{k} \frac{dT}{dt} \quad (87)$$

In order to obtain the steady-state equation, the derivative with respect to time may be written

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt}$$

and

$$\frac{\partial T}{\partial t} = 0 \quad \frac{dx}{dt} = 0$$

thus

$$\frac{dT}{dt} = \frac{\partial T}{\partial y} \quad \frac{dy}{dt} = \frac{\partial T}{\partial y} v_0 x \quad (88)$$

Combining Eqs. 87 and 88 gives

$$\frac{\partial^2 T}{\partial x^2} = \frac{x \partial T}{\partial z} \quad (89)$$

where

$$z = \frac{k}{\sigma v_0} y$$

This equation may be solved by using the Laplace transformation

$$g(x,s) = \int_0^\infty e^{-sz} T(x,z) dz \quad (90)$$

Integrating by parts gives

$$g(x,s) = -\frac{1}{s} e^{-sz} T(x,z) \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-sz} \frac{\partial T}{\partial z} dz$$

However, at $z = 0$, $T = 0$ and, at $z = \infty$, $0 = e^{-sz}$, which dominates the expression. Thus

$$sg = \int_0^\infty e^{-sz} \frac{\partial T}{\partial z} dz \quad (91)$$

Now differentiate g twice with respect to x and divide by x to obtain

$$\frac{1}{x} \frac{\partial^2 g}{\partial x^2} = \int_0^\infty e^{-sz} \frac{1}{x} \frac{\partial^2 T}{\partial x^2} dz \quad (92)$$

By subtracting Eq. 91 from Eq. 92,

$$\frac{1}{x} \frac{\partial^2 g}{\partial x^2} - sg = \int_0^\infty e^{-sz} \frac{1}{x} \frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial z} dz \quad (93)$$

However, from Eq. 89 the integral vanishes. The differential equation to be solved is

$$\frac{1}{x} \frac{\partial^2 g}{\partial x^2} = sg \quad (94)$$

The solution of this equation which does not go to ∞ at $x \rightarrow \infty$ is

$$g = A \sqrt{x} H_{1/3}^{(1)} \left(i \frac{2}{3} \sqrt{s} x^{3/2} \right) \quad (95)$$

as may be verified by differentiation. To fit the boundary condition at $x = 0$ make

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = -\frac{q}{k} \quad (96)$$

Differentiating Eq. 90 with respect to x gives

$$\frac{\partial g}{\partial x} = \int_0^\infty e^{-sz} \frac{\partial T}{\partial x} dz$$

and

$$\left. \frac{\partial g}{\partial x} \right|_{x=0} = \int_0^\infty e^{-sz} \frac{q}{k} dz = \frac{q}{ks} e^{-sz} \Big|_0^\infty = -\frac{q}{ks} \quad (97)$$

Differentiating Eq. 95 with respect to x gives

$$\frac{\partial g}{\partial x} = A x \sqrt{s} H_{(-2/3)}^{(1)} \left(i \frac{2}{3} \sqrt{s} x^{3/2} \right)$$

The limit of this expression as $x \rightarrow 0$ is

$$\left. \frac{\partial g}{\partial x} \right|_{x=0} = \frac{A i^{2/3} s^{1/6} 3^{2/3}}{\sin(\pi/3)(-2/3)!} \quad (98)$$

which by Eq. 97 is $-q/ks$. Thus A is

$$A = i^{4/3} s^{-7/6} q k^{-1} 3^{-2/3} \sin \frac{\pi}{3} \left(\frac{2}{3} \right)!$$

and Eq. 95 becomes

$$g = \sin \frac{\pi}{3} \left(-\frac{2}{3} \right)! i^{4/3} q s^{-7/6} k^{-1} 3^{-2/3} \sqrt{x} H_1^{(1)} \left(i \frac{2}{3} \sqrt{s} x^{3/2} \right) \quad (99)$$

The desired function is the temperature of the surface delivering heat, i.e., $x = 0$, putting in the $\lim_{x \rightarrow 0} \mathcal{L}_{H_1/3}^{(1)}$ gives

$$\left. g \right|_{x=0} = \frac{q 3^{-1/3} s^{-4/3}}{k(-1/3)!} \left(-\frac{2}{3} \right)! \quad (100)$$

From Eqs. 90 and 100

$$\left. g \right|_{x=0} = \int_0^\infty e^{-sz} T \Big|_{x=0} dz = \frac{q 3^{-1/3} s^{-4/3} (-2/3)!}{k(-1/3)!}$$

which may be written as

$$\int_0^\infty e^{-sz} (s^{1/3} T_0) d(s z) = \frac{q 3^{-1/3} (-2/3)!}{k(-1/3)!} = \text{constant}$$

and this is satisfied if

$$T_0 = z^{1/3} \frac{q^{3-1/3} (-2/3)!}{k(-1/3)! (1/3)!} = \frac{z^{1/3} q^{32/3}}{k(-1/3)!} \quad (101)$$

since the left-hand side becomes

$$\frac{q^{3-1/3} (-2/3)!}{k(-1/3)! (1/3)!} \int_0^\infty e^{-u} u^{1/3} du \quad u = sz$$

and the above integral defines $(1/3)!$. Q.E.D. Replacing z in Eq. 101 by its definition in Eq. 89 gives the desired function

$$T_0 = \frac{3^{2/3} q}{k^{2/3} \sigma^{1/3} v_0^{1/3} (-1/3)!} y^{1/3} \quad (102)$$

Thus the temperature increase at the heated plate, under the assumed conditions, is a constant times the cube root of the distance from the cool end to the point in question.

Since the assumed boundary conditions are not usually obtained, Eq. 102 must be used with judgement. Merely flowing enough coolant so that its average temperature rise is not excessive does not ensure that the maximum coolant temperature (i.e., the temperature of the conductor) is not excessive. It is necessary to estimate the difference between the maximum and the average. For this purpose, Eq. 102 may be used.

Laminar Flow in Triangular Duct

Westinghouse has proposed the cooling and insulating arrangement shown in Fig. 1.31, in which the coolant flows in triangular ducts. The flow in such a duct is derived below. The equation for the velocity profile is

$$\nabla^2 v + \frac{p}{\mu l} = 0 \quad (103)$$

where v = velocity

p = pressure

μ = viscosity

l = length of duct

The problem is simplified by approximating the triangular duct with the sector of a circle. Using polar coordinates Eq. 103 becomes

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{p}{\mu l} = 0 \quad (104)$$

A particular integral of this equation which satisfies the boundary conditions except along the arc (an end effect) is

$$v_1 = \frac{pr^2}{4\mu l} \left(\frac{\cos 2\theta}{\cos 2\theta_0} - 1 \right) \quad (105)$$

where θ_0 is the half angle of the sector. The homogeneous equation is solved by a series involving trigonometric functions of the angular coordinate and powers of the radius, but its contribution is small for small θ_0 [approximately 5 per cent higher pressure for a given volume of flow when $(\theta_0 \sim .02)$]. Expanding the cosines of Eq. 105 in power series and keeping only the leading term gives

$$v_1 = \frac{pr^2}{2ul} (\theta_0^2 - \theta^2) \quad (106)$$

The volume of flow per second obtained by integrating Eq. 106 is

$$V = \frac{p\rho^4 \theta_0^3}{6ul} = \frac{p\rho h^3}{4ul} \quad (107)$$

where p = outside radius of sector

$h = 2\rho\theta_0$ = base of sector

The average velocity is obtained by dividing by the area of the sector

$$\bar{v} = \frac{ph^2}{24ul} \quad (108)$$

The average one-third power of the velocity gradient is

$$v_0^{1/3} = \frac{3}{4} \left(\frac{ph}{2ul} \right)^{1/3} \quad (109)$$

It is interesting to note that the result for a rectangular duct of width h is

$$v_0^{1/3} = \left(\frac{ph}{2ul} \right)^{1/3}$$

Outside Diameter of Coils

The factors which influence the determination of the outside diameter of the coils are convenience and costs. The discussion below includes the obvious elements of cost. For operating convenience, the coils should be made smaller than is indicated because the total cost is a slowly varying function of the coil outside diameter near the minimum of cost.

In addition to knowing the type of construction to be used in the coil, it is necessary to know the performance of the magnetic circuit and the unit cost of steel, electrical energy, and coil materials. It appears that the unit costs can only be determined after the cyclotron has been in operation for several years, so estimates must be employed.

The costs which are affected are the combined costs of steel, copper, and energy.

The cost of steel may be divided into two parts: one varies linearly with the outside diameter of the coil, and the other does not depend on the outside diameter of the coil (see Fig. 1.32). The cost of steel may be expressed

$$C_{st} = C'_{st} 2\pi \frac{B}{A} S^3 R_0^2 r_0^x + \text{other costs} \quad (110)$$

In the above formula

- C'_{st} = unit cost of steel, \$/cm³
- B = area of horizontal yoke $\times 2$
- A = area of pole face
- S = width of gap, cm
- R_0 = ratio of radius of pole face to gap width
- r_0 = ratio of inside radius of coil to width of gap
- x = ratio of outside coil radius to inside coil radius
- x_0 = the value of x which indicates minimum cost

The cost of copper is given by

$$C_{cu} = C'_{cu} \pi f (x^2 - 1) S^3 r_0^2 t \quad (111)$$

where C'_{cu} = unit cost of Cu (\$/cm³)
 f = space fraction

t = ratio of height of both coils to gap width

In C'_{cu} may be included the cost of insulation and construction.

In a coil where the current is distributed as $1/r$ (the current distribution makes very little difference unless the coil OD is much greater than the coil ID), the cost of energy is given by

$$C_p = C'_p \frac{2\pi \rho S}{ft \log x} i'^2 \quad (112)$$

where C'_p = unit cost of energy (\$/10 year watt)

ρ = resistivity of conductor (ohm cm)

i' = ampere turns required if gap were 1 cm

$$i' = \frac{10 H_0}{4\pi E} \quad (113)$$

where E = efficiency of magnet = gap ampere turns + total ampere turns

H_0 = field in gap

In C'_p may be included the cost of the generator and the cost of cooling to remove the heat developed.

The sum of these costs plus the costs which are not a function of x gives the total cost of the magnet,

$$C = C'_{st} 2\pi \frac{B}{A} S^3 R_0^2 r_0 x + C'_{cu} \pi f (x^2 - 1) S^3 r_0^2 t + C'_p \frac{\rho S}{ft \log x} \frac{100 H_0^2}{8\pi E^2} + G \quad (114)$$

where G = other costs, not a function of x .

Assuming that nothing but x itself and C are functions of x in the above equation, in particular assuming E is not a function of x , Eq. 114 may be differentiated with respect to x and the minimum c determined by equating dC/dx to zero and solving for x .

$$C'_p \left(\frac{100 \rho S H_0^2}{8\pi f t E^2} \right) \frac{1}{x \log x} = C'_{st} \frac{B}{A} 2\pi R_0^2 S^3 r_0 + (C'_{cu} \pi r_0^2 f S^3 t) 2x \quad (115)$$

For purposes of computing, this form is most convenient, since constants are the same as those in the cost equation. The equation may be simplified for inspection however.

$$C'_{st} \frac{B R_0^2}{A} w x + C'_{cu} (w x)^2 = \rho C'_p \left(\frac{10 H_0}{4\pi E S} \frac{1}{\log x} \right)^2 \quad (116)$$

where $w = tr_0 f$.

It is quite clear from either equation that this factor, optimum x , depends on scale factor. The assumptions made cause the value of x given by the equation to be too large.

Equation 115 or Eq. 116 is solved by the nomograph given in Fig. 1.33. The sample calculation on the nomograph and the curve of cost vs. x given in Fig. 1.34 are based on the data:

$t = 2.357$	$H = 18,000$ gauss
$f = 0.85$	$r_0 = 4.79$
$B/A = 1.2$	$\rho = 1.88 \times 10^{-6}$ ohm cm
$E = 0.70$	$C'_p = 0.333$ \$/10 year watt
$S = 35.56$ cm	$C'_{cu} = 5.88 \times 10^{-3}$ \$/cm ³
$R_0 = 4.64$	$C'_t = 1.89 \times 10^{-3}$ \$/cm ³

Height of Coils

With a method similar to that employed in the previous paragraph, an estimate of the optimum height of the coils can be made. The terms which are of interest in the cost equation are

$$\left. \begin{aligned} & C'_{st} \pi R_0^2 S^3 \left[\left(\frac{R_B}{R_0} \right)^2 + \frac{B^1}{A} \right] (F - k') \\ & + C'_{cu} 2\pi R_0^2 S^3 \left[(x^2 - 1) \frac{r_0^2}{R_0^2} (F - k'') \right] \\ & + C'_p \frac{\rho S}{t(1 - \delta) \log x} \frac{(10H_0 M)^2}{8\pi} + G' \end{aligned} \right\} \quad (117)$$

Most of the terms have been defined previously. The others are as follows:

- G' = other costs
- $2SF$ = height of vertical yokes; K' , K'' are constants
- $B' = 2 \times$ area of vertical yoke
- δ = that fraction of the coil volume not occupied by conductor
- SR_0 = radius of base of pole
- $M = \frac{1}{E}$ = ratio of total ampere turns to gap ampere turns

The flux density in the base of the pole is ordinarily so high that the variation of M or E with F is important. Considering this, the differentiation with respect to F gives

$$2\pi R_0^2 S^3 \left\{ C'_{st} \left[\left(\frac{R_B}{R_0} \right)^2 + \frac{B'}{A} \right] + C'_{cu} \left[(x^2 - 1) \left(\frac{r_0}{R_0} \right)^2 \right] \right\} + C'_p \frac{2\rho S}{\log x} \frac{100H_0^2}{8\pi} \left[-\frac{k^2}{(t - \delta)^2} + \frac{k}{(t - \delta)} \frac{dk}{df} \right] = 0 \quad (118)$$

as the condition for the optimum coil height.

For example, the data

$C'_p = 0.441$ \$	$H_0 = 20,000$ gauss
$C'_{st} = 1.9 \times 10^{-3}$ \$/cm ³	$R_0 = 4.64$
$C'_{cu} = 0.007$ \$/cm ³	$B/A = 1.1$
$S = 35.56$ cm	$R_B/R_0 = 144/130$
$\rho = 1.75 \times 10^{-6}$ ohm cm	$r_0/R_0 = 146/130$
$x = 1.4$	$M = 1.47$

and $dk/df = 0.129$, corresponding to $H = 1290$ in the pole base give as the optimum, $S(t - \delta)/2 = 16\frac{2}{3}$ -in. = height of conductor.

Other Factors in Coil Design

There is a tendency to overestimate the quality of the connection which is required between two large conductors in a coil. The connection must not develop hot spots, which would deteriorate insulation, and should have a small resistance and good mechanical strength. One square centimeter of metallic contact distributed over the joint will carry 10,000 amps with negligible resistance and temperature rise. Adequate mechanical strength is almost a sufficient requirement for the connection.

The generator should match the coil in the sense that the quotient of the maximum voltage output and the maximum current should be equal to the resistance of the coil.

For small magnets where the coil must be relatively large to keep the power requirement reasonable, the distribution of current in the coil may be important. The resistance per ampere turn squared is a minimum when the current density is inversely proportional to the radius. This is not necessarily the best distribution for a magnet since the leakage flux is a function of the ampere-turn distribution.

The whole problem of coil design is greatly complicated by lack of industrial experience. This makes it difficult to determine the cost of construction in advance, and it is not possible to get reasonable estimates on the relative cost of two designs. In the future, it will be possible to include intelligently the cost of construction as a parameter in coil design.

1.3 POLE BASE, YOKE, AND OPTIMUM FIELD

Introduction

Steel plays three important roles in a cyclotron magnet. It shapes the field in the gap, completes the flux circuit in a material of low reluctance, and is the principal structural member of the magnet.

The structural problems are not considered here except to note that the critical dimensions of the gap can be maintained by nonmagnetic supports at the edge of the gap.

As was described in Sec. 1.1, steel is useful in giving the field in the gap the proper dependence on position. It is also important in reducing the ampere turns required to complete the magnetic circuit from one pole tip to the other. Since steel is costly and does not have infinite permeability, the circuit should be kept as short as possible. The area of the pole base and yoke sections should be adjusted so that the over-all cost of the magnet is a minimum. Sections that are too large require excessive steel and sections that are too small require excessive power.

The magnetic circuit is kept short by making the yoke sections wide and thin so that a given inside circumference is maintained with a minimum average circumference. Also, the outside, minus the inside radius of the coils, divided by the sum of the axial dimensions of the two coils, should be approximately equal to one so that the circumference of a vertical cross section of the coils (thus the circumference of the steel circuit) is made a minimum. Both the above statements need qualification and are useful only as guides.

The best cross-sectional area for the pole base and yoke is estimated in the following paragraphs. Since the size of the coils and horizontal yokes are affected by the size of the pole base, it should be made of magnetically good steel, and the optimum size is rather critical. The space occupied by the steel in the yoke does not influence the accessibility of the gap or the size of the coil; therefore, the quality of steel used in this part of the magnet is less important. The experimental adjustment of the yoke can be changed and the arrangement which gives the minimum cost of steel plus a high ratio of power can be selected. For this reason, the problem is treated in a very simple way. The results may be used as a starting point for the experimental work.

In the subsection, Optimum Field, a method is outlined for determining the field strength at which the magnet should operate so as to give the required energy at the least cost. This field strength depends on the design and the size of the magnet and cannot be determined without a model magnet.

Size of Pole Base

The size of the pole base can be approximately adjusted by the following analytic procedure so as to give a minimum over-all cost.

The costs that are a function of the diameter of the pole base are the combined costs of horizontal yokes, pole steel, steel rings (which form base plates for the coil and act as part of the magnetic circuit), coil, and energy to operate the coil. Assuming that the cost of the steel is proportional to its volume, that the cost of the coil is proportional to the copper volume, and that the cost of energy is proportional to the resistance of the coil times the square of the current it carries, gives as the sum of these costs

$$\begin{aligned}
& C'_{st} \left[2\pi S^2 R_0^2 \frac{D}{A} S r_0 x + \pi S^2 R_B^2 S l_B + \pi S^2 R_B^2 (x^2 - 1) SG \right] \\
& + C'_{cu} \left[\pi S^2 r_0^2 (x^2 - 1) St f \right] \\
& + C'_p \left[\frac{\Omega}{S} \left(\frac{10 S H_0 M}{4\pi} \right)^2 \right]
\end{aligned} \tag{119}$$

where C'_{st} , C'_{cu} , and C'_p are cost of steel per cubic centimeter, copper per cubic centimeter, and energy per 10 year watt.

S = gap width, cm

SR_0 = radius of pole face

$S^2 D = 2 \times$ area of horizontal yoke

$S^2 A = \pi S^2 R_0^2$ = area of pole face

Sr_0 = inside radius of coil

x = outside coil radius divided by inside coil radius

SR_B = radius of pole base

$Sl_B = 2 \times$ height of pole base

$SG = 2 \times$ thickness of steel base plate for coil

$St = 2 \times$ height of coil

f = fraction of coil volume occupied by copper

Ω/S = resistance of coils (in series) per turn squared

H_0 = field at center of gap

M = ampere turns required by magnet divided by ampere turns used in gap

i_g = ampere turns used in gap = $\left(\frac{10 S H_0}{4\pi} \right)$

i_B = ampere turns used in pole base

The differentiation of this expression with respect to R_B , in order to minimize the cost, is straightforward (assuming $r_0 = R_B + \text{constant}$) except for the dependence of M on R_B ,

$$\begin{aligned}
& 2\pi S^3 C'_{st} \left[R_0^2 \frac{D}{A} x + R_B l_B + R_B (x^2 - 1) G \right] \\
& + 2\pi S^3 C'_{cu} r_0 (x^2 - 1) t f + 2 S C'_p \Omega \left(\frac{10 S H_0}{4\pi} \right)^2 M \frac{dM}{dR_B}
\end{aligned} \tag{120}$$

M may be written

$$M = \text{const} + \frac{i_B}{i_g} = \text{const} + \frac{1_B H_B}{H_0}$$

and differentiating

$$\frac{dM}{dR_B} = \frac{1_B}{H_0} \frac{dH_B}{dB_B} \frac{dB_B}{dR_B} \tag{121}$$

Now assuming the flux ϕ_0 in the pole base is a constant

$$\frac{dB_B}{dR_B} = \frac{d}{dR_B} \frac{\phi_0}{\pi R_0^2} = -\frac{2}{R_B} \frac{\phi}{\pi R_0^2} = -\frac{2B_B}{R_B} \tag{122}$$

By putting Eqs. 121 and 122 in Eq. 120 and equating it to zero, a condition results which will make the over-all cost of the magnet a minimum. Solving this equation for $B/(dB/dH)$, an observable property of the steel, gives

$$\frac{B_B}{dB_B/dH_B} = \frac{C'_{st} \left[R_0^2 \frac{D}{A} x + R_B^2 B + R_B(x^2 - 1)G \right] + C'_{cu} r_0(x^2 - 1)tf}{100 \frac{C'_p}{S^2} \frac{\Omega H_0}{8\pi^3} \frac{M^2 B}{R_B}} \quad (123)$$

where, for the usual type of coil,

$$\Omega = \frac{\pi \rho}{tf} \left(\frac{x+1}{x-1} \right) \quad (124)$$

ρ = resistivity of conductor, ohm cm

$B/(dB/dH)$ varies rapidly with B near the saturation density of the steel, as is indicated by the following table based on a low-carbon steel sample tested by the procedure outlined in the following chapter.

B	$B/(dB/dH)$
20,880	6.8×10^3
21,020	8.4×10^3
21,200	10.6×10^3

Due to the rapid change in $B/(dB/dH)$, the optimum flux density in the pole base does not change appreciably when small changes in the design are made; however, the factors not considered, such as air leakage and the convenience of having a small machine, make the flux density indicated by Eq. 123 too small.

The following example has data appropriate for a 130-in.-diam. magnet with a 14-in. gap.

$$\begin{aligned} \Omega &= 1.44 \times 10^{-5} \text{ ohm cm per (turn)}^2 & D/A &= 1.2 \\ C'_{st} &= 0.0019 \text{ \$/cm}^3 & x &= 1.442 \\ C'_{cu} &= 0.007 \text{ \$/cm}^3 & l_B &= 1.9 \\ C'_p &= 0.44 \text{ \$/10 year watt} & G &= 0.5 \\ S &= 35.56 \text{ cm} & tf &= 2.19 \\ R_0 &= 4.64 & H_0 &= 2 \times 10^4 \text{ gauss} \\ R_B &= 5.23 & M &= 1.43 \\ r_0 &= 5.30 \end{aligned}$$

These figures give 8.3×10^3 oersteds for $B/(dB/dH)$ which, as can be seen from the table, corresponds to about $B = 21,000$ gauss in low-carbon steel.

Yoke

As a simple approximation, the yoke is assumed to be constructed of rectangular parallelepipeds of steel. The horizontal yokes are assumed to carry all the flux from the pole, and the air path is assumed to have a reluctance which is in parallel with that of the vertical yokes. The equivalent magnetic circuit is shown in Fig. 1.35. For this approximation, the costs involved for the horizontal yokes are the cost of energy for the coil, plus the volume of steel in horizontal yokes, times the cost per unit volume. Assuming that the yokes have a constant area $(S^2 D)/2$ over their length KS , these costs are

$$C'_p \frac{\Omega}{S} \left(\frac{10SH_0 M}{4\pi} \right)^2 + C'_{st} \pi R_0^2 S^2 KS \frac{D}{A} \quad (125)$$

The derivative of this expression with respect to D multiplied by D is

$$C'_p D \frac{\Omega}{S} \left(\frac{10SH_0 M}{4\pi} \right)^2 2M \frac{dM}{dD} + C'_{st} \pi R_0^2 S^3 K \frac{D}{A} \quad (126)$$

In order to determine the derivative of M with respect to D , assume that the field in the horizontal yokes is H_H , a constant, over the whole length K . Then

$$\frac{dM}{dD} = \frac{1}{i_g} \frac{d(i_0 + i_H)}{dD} = \frac{k}{H_0} \frac{dH_H}{dB_H} \frac{dB_H}{dD} \quad (127)$$

where i_0 = ampere turns required by remainder of circuit (constant)

i_g = ampere turns required by gap

i_H = ampere turns required by horizontal yoke

B_H = flux density in horizontal yoke

Assuming that the flux ϕ is constant throughout the length of the yoke, then

$$\frac{dB_H}{dD} = \frac{d\phi}{dD D} = -\frac{B_H}{D} \quad (128)$$

By putting Eqs. 127 and 128 into Eq. 126 and equating the result to zero, the condition for minimum cost is found to be

$$C_p \frac{\Omega}{S} \left(\frac{10SH_0}{4\pi} \right)^2 \frac{2M}{H_0} \left(\frac{B_H}{dB_H/dH_H} \right) - C_{st} \pi S^3 R_0^2 \frac{D}{A} \quad (129)$$

Solved for $B_H/(dB_H/dH_H)$, this is

$$\frac{B_H}{dB_H/dH_H} = \left(\frac{4\pi S}{10} \right)^2 \frac{\pi R_0^2 (D/A)}{2MH_0\Omega} \frac{C_{st}}{C_p} \quad (130)$$

Using the data from the previous paragraph, gives for $B/(dB/dH)$ the value of 850 oersteds, which corresponds to a flux density of less than 18,000 gauss in low-carbon steel.

The area of the vertical yokes can be calculated in a similar manner, again using the equivalent circuit shown in Fig. 1.35. The costs involved are the cost of energy for the coil, plus the volume of steel in the vertical yokes, times the cost per unit volume. Assuming that the yokes have a constant area $(S^2 D')/2$ over their height Sh these costs are

$$C_p \frac{\Omega}{S} \left(\frac{10SH_0 M}{4\pi} \right)^2 + C_{st} \pi R_0^2 S^2 h S \frac{D'}{A} \quad (131)$$

The derivative of this expression with respect to D' multiplied by D' is

$$D' C_p \frac{\Omega}{S} \left(\frac{10SH_0}{4\pi} \right)^2 2M \frac{dM}{dD'} + C_{st} \pi R_0^2 S^3 h \frac{D'}{A} \quad (132)$$

The derivative of M with respect to D' is

$$\frac{dM}{dD'} = \frac{h}{H_0} \frac{dH_v}{dD'} \quad (133)$$

where H_v = the field in the vertical yokes. In order to get this expression in useful form, the equation for the flux in the circuit,

$$\phi = B_v D' + \frac{h H_v}{k} \quad (134)$$

where B_v = flux density in the vertical yokes and k/S is a measure of the reluctance of the air path, is differentiated with respect to D' .

$$0 = B_v + D' \frac{dB_v}{dH_v} + \frac{h}{k} \frac{dH_v}{dD'} \quad (135)$$

The equation for minimum cost is obtained by substituting Eq. 135 and 133 in Eq. 132, and setting this equal to zero. When this equation is solved for $B/(dB/dH)$, the result is

$$\frac{B_v}{dB_v/dH_v} \left(\frac{C'_p \Omega H_0 (10/4\pi S)^2 2M}{C_{st} \pi R_0^2 (D'/A)} - \frac{h}{B_v k (D'/A) \pi R_0^2} \right)^{-1} \quad (136)$$

Assuming the data used in the preceding examples, and also $B_v = 18,000$, $h = 5$, $k = 10^{-2}$, and $D'/A = 1.0$, gives for $B_v/(dB_v/dH_v)$ the figure 1300 oersteds. This corresponds to about $B_v = 18,000$ gauss in low-carbon steel.

Optimum Field

There is an optimum field strength at which to operate the cyclotron magnet. That is, in order to obtain a given energy, it may be better to build a large magnet and operate it at a low field or a smaller magnet and operate it at a higher field strength.

The dependence of the cost of a cyclotron on its size may be written as

$$C = C'_3 S^3 + C'_2 S^2 + C'_1 S + C_0 \quad (137)$$

where $C_3 = C'_3 S^3$ = costs that increase with the cube of the magnet size (e.g., steel)

$C_2 = C'_2 S^2$ = costs that go with the square (e.g., cyclotron shield)

$C_1 = C'_1 S$ = costs that go with the first power (e.g., power)

C_0 = costs that do not depend on the size of the magnet

The derivative with respect to size, S , times S is

$$S \frac{dC}{dS} = 3C_3 + 2C_2 + C_1 \quad (138)$$

The relation between energy and size is approximately

$$E = E' S^2$$

and

$$S \frac{dE}{dS} = 2E \quad (139)$$

The quotient of Eqs. 138 and 139 is

$$\frac{dC}{dE} = \frac{3C_3 + 2C_2 + C_1}{2E} \quad (140)$$

The dependence of magnet cost on exciting current in the coils is

$$C = k_1 i^2 + \text{other costs}$$

where $C_p = k_1 i^2$ = cost of energy to operate the coils.

The derivative of the cost is

$$dC = 2k_1 i di = 2C_p \frac{di}{i} \quad (141)$$

The energy of the beam may be written approximately as

$$E = k_2 (H\rho)^2 \quad (142)$$

and

$$dE = 2k_2 H\rho d(H\rho) = 2E \frac{d(H\rho)}{H\rho} \quad (143)$$

The quotient of Eqs. 142 and 143 is

$$\frac{dC}{dE} = \frac{C_p}{E} \frac{d(\log i)}{d(\log H\rho)} \quad (144)$$

When the increase in cost per unit of beam energy is the same due to increasing the field as it is due to increasing the scale, then the magnet is operating at the optimum field strength. By setting Eq. 140 equal to Eq. 144 gives the condition,

$$\frac{d(\log i)}{d(\log H\rho)} = \frac{3C_3 + 2C_2 + C_1}{2C_p} \quad (145)$$

The quantity on the left must be determined experimentally on a magnet that is close to its final form.

This concludes the discussion of the analytic design of a cyclotron magnet. It is clear that the final design must be based on a working model, but this work should be a valuable guide.

1.4 EXPERIMENTAL METHODS

Introduction

In experimental design work, it is necessary to have methods for obtaining data and methods for using these data. The following discussion includes a few methods for obtaining data by using a long-period galvanometer, arranged so that its net deflection is zero when the circuit is balanced, and two methods of using field-shape data in adjusting the shim.

Basic Circuit

Once the flux or flux density is known at any point in the magnet, the remaining data on flux distribution may be measured by determining the ratio of two quantities of flux. In most cases, the ratio is not so sensitive to the current used to excite the magnet as the corresponding absolute value and the required accuracy of current control is reduced.

The circuit which is employed in connection with the following methods consists of a galvanometer wired as shown in Fig. 1.36.

The deflection of the galvanometer, if it moves from a position of rest to a position of rest, is

$$\Delta\theta = k \left(\frac{\Delta\phi_1 r_1}{r_1 + R_1} - \frac{\Delta\phi_2 r_2}{r_2 + R_2} \right) \quad (146)$$

assuming that the galvanometer is linear and that its restoring torque is zero. The meaning of the symbols is

- $\Delta\theta$ = deflection of galvanometer
- k = constant for a given galvanometer
- $\Delta\phi$ = change in flux-linking search coil
- r = resistance which is common to search coil and galvanometer circuits
- R = resistance in search-coil circuit (including leads and coil)
- Ω = resistance in galvanometer circuit

This equation is correct regardless of series inductances or parallel capacitances which may be in the circuit (provided the flux linkages of the inductances return to the same value at zero current and the net charge in the condensers returns to the same value at zero voltage).

Where a null method is employed, in which the galvanometer does not move in the balanced condition, the assumption that the restoring torque is zero is of no consequence, and if the galvanometer is not dynamically balanced, its motion can be steadied in the off-balance condition by increasing Ω . However, where there is a difference in phase between the changes in flux in the two search coils, the restoring torque becomes important. In this case the period of the galvanometer must greatly exceed the difference in phase between the two flux changes. This is accomplished by using a galvanometer which has a large time constant and by keeping the resistance, into which the galvanometer works, small (i.e., keeping Ω , and others, small). When $\Delta\theta$ is zero, Eq. 146 gives

$$\frac{\Delta\phi_1}{\Delta\phi_2} = \frac{r_2}{r_1} \frac{r_1 + R_1}{r_2 + R_2} \quad (147)$$

The errors in the null methods are due to (1) insufficient sensitivity, corrected by using search coils with more turns-area; (2) incorrect resistances, including such masked effects as changes in the resistance of the search coils due to temperature changes; (3) incorrect or insufficient knowledge about the geometry and location of the search coils; (4) restoring torque in the galvanometer, if the galvanometer moves when the circuit is balanced; and (5) incorrect or insufficient knowledge of the current exciting the magnet being studied. (Also the history of the exciting current is important where hysteresis effects are important.)

Comparing Turns-area of Two Search Coils

A particularly simple example of the use of the preceding circuit is the comparison of the turns-area of two search coils. If the two coils are placed in a homogeneous field with their axes accurately aligned, and the resistances r_1 , R_1 , r_2 , and R_2 adjusted so that the galvanometer does not deflect when the field is changed, then Eq. 147 gives

$$\frac{\Delta\phi_1}{\Delta\phi_2} = \frac{\Delta H_1 A_1}{\Delta H_2 A_2} = \frac{r_2}{r_1} \left(\frac{r_1 + R_1}{r_2 + R_2} \right) \quad (148)$$

or

$$\frac{A_1}{A_2} = \frac{r_2}{r_1} \left(\frac{r_1 + R_1}{r_2 + R_2} \right)$$

where A = turns-area of search coil

This operation is simple and capable of good accuracy.

Permeability and Incremental Permeability

Permeability and incremental permeability can be measured by comparing a quantity of flux proportional to H with a quantity proportional to B .

The flux in a coil wound on the sample changes by an amount

$$\Delta\phi_1 = a(\Delta H) + A(\Delta B) \quad (149)$$

where A = turns in coil times area of sample

a = turns-area of coil minus turns in coil times area of sample

H = field strength

B = flux density in sample

The term proportional to ΔH may be negligible; if not, it can be eliminated by a compensating coil so that $\Delta\phi_1$ is proportional to B . A second coil arranged so that it does not link the sample links the flux

$$\Delta\phi_2 = b(\Delta H) \quad (150)$$

where b = turns-area of the field coil.

These coils are connected to the galvanometer as described in the subsection, Basic Circuit. If the galvanometer does not deflect when the field is changed, then

$$\frac{\Delta\phi_1}{\Delta\phi_2} = \frac{r_2}{r_1} \left(\frac{r_1 + R_1}{r_2 + R_2} \right) = \frac{A(\Delta B)}{b(\Delta H)} \quad (151)$$

The significance of $\Delta B/\Delta H$ depends on the history of H in the sample. If the sample is cycled, the permeability is measured. If the field varies slowly in one direction, the deflection of the galvanometer will go through a maximum when the differential permeability is that indicated by Eq. 151. At high fields, this differs but little from the incremental permeability. The incremental permeability may be measured by superimposing a sufficiently low-frequency alternating field on a steady field and adjusting the resistances r_1 , R_1 , r_2 , and R_2 so that the galvanometer does not deflect.

Field-shape Measurement

The dependence of the field, in the mid-plane of the gap, on position is called the field shape. The field shape determines the focusing conditions for the ion beam and the useful radius of the field. Some of the difficulties associated with determining the field shape by measuring the absolute field at various points can be avoided by comparing the field at the point in question with the field at the center of the mid-plane.

The apparatus designed for this purpose consists of two coils mounted with their axes parallel to each other and at right angles to the axis of a shaft. The axis of this shaft lies in the mid-plane of the gap and passes through the center of the gap. The shaft is made to rotate through 180 deg and then back with a frequency of about 1 cycle per second. The coils start with their axes parallel to the field, are inverted, and then returned to their original position. This type of motion is used to avoid the necessity of having a commutator.

The signal due to one coil may be written

$$\Delta\phi_1 = k_1 H_1 \cos \theta \quad (152)$$

where k_1 = constant

H_1 = field at the coil

θ = angular position of shaft

If the alignment of the coils is not perfect, there may be a difference in phase δ in the signal from the second coil,

$$\Delta\phi_2 = k_2 H_2 \cos (\theta + \delta) \quad (153)$$

Since δ is a small angle, this may be written

$$\Delta\phi_2 = k_2 H_2 (\cos \theta - \delta \sin \theta) \quad (154)$$

The term in $\sin \theta$ is a nuisance and is eliminated by a third coil whose axis is at right angles to the axis of the shaft and the axes of the other two coils. The arrangement of the coils and the circuit employed is shown in Fig. 1.37.

The apparatus is used by placing coils 1 and 2 in equal fields (e.g., at the same distance from the center of the mid-plane) and adjusting R_1 , r_1 , R_2 , and r_2 so that a rectified sine wave is produced by the galvanometer. R_3 and r_3 are then adjusted so that the amplitude of the sine wave is zero, and the galvanometer remains stationary. In order that this adjustment need only be made once, all of the coils are rigidly connected to the shaft. This means that the fields at a fixed radial distance are compared, and a certain amount of interpolation is necessary in order to determine the field shape. The resistances R_2 and r_2 are so arranged that their sum is a constant, and the values are so chosen that the ratio of the field at coil 1 to the field at coil 2 is read directly when the galvanometer shows no deflection.

By this arrangement, an accuracy of a few hundredths of 1 per cent has been obtained. In order to achieve this accuracy, the search coils must be carefully constructed. Two methods for constructing accurate search coils are discussed in Appendix IV.

Magnet Efficiency and Other Problems

Another use for the basic circuit shown in Fig. 1.36 is the measurement of the efficiency of the magnet. The efficiency is defined as the ratio of the ampere turns required by the gap to the total ampere turns required by the magnet. Since the ampere turns required by the gap are proportional to the field in the gap, a change in flux proportional to the gap ampere turns may be obtained by a mutual inductance with its primary in the magnet coil circuit. These signals may be compared by the null method outlined in the subsection, Basic Circuit, and their ratio, which is proportional to the efficiency, can be obtained.

Other uses for this circuit will occur to the user as the information is required. For example, the flux density at various points in the pole and yoke can be determined by comparing the flux linking a coil placed around the region being investigated with the flux linking a coil in the gap.

Numerical Calculation of Potential off Mid-plane

The nature of the field is important for some distance off the mid-plane, because the ion beam does not confine itself to this plane. There are several ways of determining the shape of the equipotentials off the mid-plane. One is, of course, the direct measurement of the potential. This is difficult, but not impossible, in a model magnet. An apparatus similar to that described in the subsection, Field-shape Measurement, equipped with potential measuring coils, could be used.

The potential off the mid-plane can be determined in principle by sufficiently accurate and detailed knowledge of the field in the mid-plane. These data may be used in the numerical procedure outlined below. The two-dimensional method employed is quite satisfactory, if the potential at points near the center of the magnet is not required.

The space in which the potential is to be calculated is divided by equally spaced lines which are parallel to the axes (see Fig. 1.38). The intersections of these lines form a mesh. The points in this mesh are indexed by two subscripts which specify the coordinates of the point; the index m specifies the distance from the mid-plane and the index n , the radial position.

The potential in the mid-plane may be taken as zero, thus

$$V_{0,n} = 0$$

for all n , and the potential at unit distance off the mid-plane is clearly

$$V_{1,n} = H_n$$

where H_n = field in the mid-plane at position 0,n measured in suitable units. It may be shown that the potentials for higher m may be written as

$$\begin{aligned}
 V_{2,n} &= 4(H_n) - (H_{n-1} + H_{n+1}) \\
 V_{3,n} &= 17(H_n) - 8(H_{n-1} + H_{n+1}) + (H_{n-2} + H_{n+2}) \\
 V_{4,n} &= 80(H_n) - 49(H_{n-1} + H_{n+1}) + 12(H_{n-2} + H_{n+2}) - (H_{n-3} + H_{n+3}) \\
 V_{5,n} &= 401(H_n) - 280(H_{n-1} + H_{n+1}) + 97(H_{n-2} + H_{n+2}) - 16(H_{n-3} + H_{n+3}) + (H_{n-4} + H_{n+4}) \\
 V_{6,n} &= 2084(H_n) - 1569(H_{n-1} + H_{n+1}) + 672(H_{n-2} + H_{n+2}) - 161(H_{n-3} + H_{n+3}) \\
 &\quad + 20(H_{n-4} + H_{n+4}) - (H_{n-5} + H_{n+5}) \\
 V_{7,n} &= 11,073(H_n) - 8752(H_{n-1} + H_{n+1}) + 4321(H_{n-2} + H_{n+2}) - 1320(H_{n-3} + H_{n+3}) \\
 &\quad + 241(H_{n-4} + H_{n+4}) - 24(H_{n-5} + H_{n+5}) + (H_{n-6} + H_{n+6})
 \end{aligned}$$

It is clear from the coefficients which enter into the formulas for $V_{m,n}$ for large m that the data must be very good to justify calculating these potentials. This method might be useful in determining the potential of the steel surface in order to correct the shim shape or to determine whether the field is such that the ions are focused over the whole height of the beam.

Adjusting Shim Near Edge

The solution to the shim-edge and pole-side problem, given in Sec. 1.1 merely suggests a shape for the shim. In order to actually determine the required shape, a cut and try procedure must be used. The convergence of this procedure may be speeded by employing the following system.

First, determine the field shape, then have steel removed below the region where the field is too high. Determine the field shape with the new shim shape. The difference between the two field-shape curves should then be graphed. The area A under this curve should correlate with the area a of steel removed (in a vertical cross section) according to the equation

$$\frac{A}{H_0 S} = K_1 \frac{a}{S^2}$$

where H_0 = field in the center of the gap

S = half-gap

The value of K_1 is somewhat sensitive to the place where the steel is removed. The average value is about 0.6.

If the steel is removed from a region whose radial width is small in comparison with the half-gap of the magnet, there is a correlation between the maximum amplitude of the field removed and the area of steel removed.

$$\frac{\Delta H}{H_0} = K_2 \frac{a}{S^2}$$

The value of K_2 depends on the place from which the steel is taken and averages 0.6. The maximum reduction in the field occurs approximately over the place from which the steel is removed, and the reduction in the field extends about a half-gap either way from the maximum.

Accuracy in the field-shape measurements and in the shop work are of great assistance in obtaining good values for the constants K_1 and K_2 with small test cuts.

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APPENDIX I

SMALLEST ROOT OF QUADRATIC EQUATION

Table 1.1 is designed to solve the equation

$$ax^2 + bx + c = 0$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (155)$$

for the root which has the smallest magnitude, when the condition

$$\Delta = \frac{ac}{b^2} < 0.0500 \quad (156)$$

is satisfied. This root is difficult to determine, because the two terms in the numerator of Eq. 155 are approximately equal. Substituting Eq. 156 into 155 gives

$$-\frac{xb}{c} = \frac{1}{2\Delta} (1 - \sqrt{1 - 4\Delta}) \quad (157)$$

A series expansion of the radical in Eq. 157 gives

$$-\frac{xb}{c} = 1 + g = 1 + \Delta + 2\Delta^2 + 5\Delta^3 + 14\Delta^4 + 42\Delta^5 + \dots \quad (158)$$

This series was used to make the table, which is presumably not wrong by more than one in the last figure.

VALUE OF FUNCTION f/x^2 APPEARING IN SHIM-SHAPE CALCULATIONS

The second table gives values of $[f(x)]/x^2$ which appears in the equation for the potential of a symmetric field which decreases linearly with increasing radius (see Eq. 3). The function f/x^2 has been computed for values of x from 0.08 to 500. The values from 0.08 through 1.00 were computed by the series,

$$\frac{f}{x^2} = \frac{1}{x} - \frac{x}{3!} + \frac{x^3}{5!} - \frac{3^2 x^5}{7!} + \frac{3^2 \times 5^2 x^7}{9!} - \dots \quad (159)$$

and from 1.00 through 500 by the equation

$$\frac{f}{x^2} = \frac{1}{4x^2} [3x \sqrt{x^2} + 1 + (1 - 2x) \sinh^{-1} x] \quad (160)$$

Table 1.1—Smallest Root of Quadratic Equation*

g			g			g		
Δ	$\Delta > 0$	$\Delta < 0$	$ \Delta $	$\Delta > 0$	$\Delta < 0$	$ \Delta $	$\Delta > 0$	$\Delta < 0$
0.0000	0.00000	-0.00000	0.0175	0.01814	-0.01691	0.0350	0.03769	-0.03275
0.0005	0.00050	-0.00050	0.0180	0.01868	-0.01738	0.0355	0.03827	-0.03318
0.0010	0.00100	-0.00100	0.0185	0.01922	-0.01785	0.0360	0.03885	-0.03362
0.0015	0.00151	-0.00150	0.0190	0.01976	-0.01832	0.0365	0.03944	-0.03406
0.0020	0.00201	-0.00199	0.0195	0.02030	-0.01878	0.0370	0.04002	-0.03449
0.0025	0.00252	-0.00249	0.0200	0.02084	-0.01924	0.0375	0.04061	-0.03493
0.0030	0.00302	-0.00298	0.0205	0.02139	-0.01971	0.0380	0.04119	-0.03536
0.0035	0.00353	-0.00348	0.0210	0.02193	-0.02017	0.0385	0.04179	-0.03579
0.0040	0.00403	-0.00397	0.0215	0.02248	-0.02063	0.0390	0.04237	-0.03622
0.0045	0.00454	-0.00446	0.0220	0.02303	-0.02109	0.0395	0.04296	-0.03665
0.0050	0.00505	-0.00495	0.0225	0.02357	-0.02154	0.0400	0.04356	-0.03709
0.0055	0.00556	-0.00544	0.0230	0.02412	-0.02200	0.0405	0.04415	-0.03752
0.0060	0.00607	-0.00593	0.0235	0.02467	-0.02246	0.0410	0.04475	-0.03794
0.0065	0.00659	-0.00642	0.0240	0.02522	-0.02292	0.0415	0.04535	-0.03837
0.0070	0.00710	-0.00690	0.0245	0.02578	-0.02338	0.0420	0.04595	-0.03880
0.0075	0.00762	-0.00739	0.0250	0.02634	-0.02383	0.0425	0.04654	-0.03922
0.0080	0.00814	-0.00788	0.0255	0.02689	-0.02428	0.0430	0.04715	-0.03966
0.0085	0.00865	-0.00836	0.0260	0.02745	-0.02474	0.0435	0.04776	-0.04008
0.0090	0.00917	-0.00885	0.0265	0.02801	-0.02519	0.0440	0.04836	-0.04051
0.0095	0.00969	-0.00933	0.0270	0.02857	-0.02564	0.0445	0.04897	-0.04094
0.0100	0.01021	-0.00980	0.0275	0.02913	-0.02609	0.0450	0.04957	-0.04135
0.0105	0.01073	-0.01029	0.0280	0.02969	-0.02645	0.0455	0.05018	-0.04178
0.0110	0.01125	-0.01077	0.0285	0.03025	-0.02699	0.0460	0.05079	-0.04221
0.0115	0.01178	-0.01125	0.0290	0.03082	-0.02744	0.0465	0.05139	-0.04262
0.0120	0.01230	-0.01172	0.0295	0.03138	-0.02788	0.0470	0.05202	-0.04304
0.0125	0.01283	-0.01220	0.0300	0.03195	-0.02833	0.0475	0.05263	-0.04346
0.0130	0.01335	-0.01267	0.0305	0.03252	-0.02878	0.0480	0.05324	-0.04388
0.0135	0.01388	-0.01315	0.0310	0.03308	-0.02922	0.0485	0.05386	-0.04429
0.0140	0.01441	-0.01363	0.0315	0.03366	-0.02965	0.0490	0.05448	-0.04472
0.0145	0.01494	-0.01410	0.0320	0.03424	-0.03010	0.0495	0.05510	-0.04512
0.0150	0.01547	-0.01457	0.0325	0.03481	-0.03054	0.0500	0.05573	-0.04555
0.0155	0.01600	-0.01504	0.0330	0.03538	-0.03098			
0.0160	0.01653	-0.01551	0.0335	0.03596	-0.03143			
0.0165	0.01707	-0.01598	0.0340	0.03653	-0.03187			
0.0170	0.01761	-0.01645	0.0345	0.03711	-0.03231			

*In order to determine the smallest root of the quadratic equation,

$$ax^2 + bx + c = 0$$

compute

$$\Delta = \frac{ac}{b^2}$$

and determine g from the table. Then $x = -(1 + g) \frac{c}{b}$ is the root.

Table 1.2—Value of Function f/x^2 Appearing in Shim-shape Calculations*

x	f/x^2	x	f/x^2
0.08	12.487	1.00	0.8403
0.09	11.096	1.20	0.6448
0.10	9.983	1.40	0.4979
0.12	8.313	1.60	0.3819
0.14	7.120	1.80	0.2869
0.16	6.223	2.00	0.2071
0.18	5.525	2.50	0.0501
0.20	4.967	3.00	-0.0681
0.25	3.958	4.00	-0.2415
0.30	3.284	5.00	-0.3682
0.40	2.434	7.00	-0.5509
0.50	1.918	10.00	-0.7387
0.60	1.568	15.00	-0.9448
0.70	1.314	20.00	-1.0914
0.80	1.121	30.00	-1.2956
0.90	0.9663	40.00	-1.4401
1.00	0.8403	60.00	-1.6433
		100.00	-1.8990
		150.00	-2.102
		200.00	-2.246
		300.00	-2.448
		500.00	-2.704

*A table of values of the function

$$f = \frac{1}{4} \left[3x \sqrt{x^2 + 1} + (1 - 2x) \sinh^{-1} x \right]$$

APPENDIX II

CALCULATION OF A THREE-DIMENSIONAL MAGNET

Introduction

The method developed in the following pages may be used to calculate the field produced by a magnet which has a broad yoke, cylindrical poles, and is energized by thin coils which are close to the steel. It gives excellent results when the flux density is low.

The geometry assumed is shown in the Fig. 1.39. The poles are taken as right circular cylinders and the yokes as infinite planes. The unit of length is the width of the half-gap.

Rotational symmetry about the axis $r = 0$ and symmetry of reflection in the plane $z = 0$ are assumed. This geometry is similar to that found in many cyclotron magnets. The errors introduced by the assumption of infinite permeability depend on the magnet design. The assumption that the energizing coil is very thin and lies on the steel surface is not very satisfactory in general.

This type of calculation should be useful in preliminary work on the design of a magnet, mainly to get an approximate value for the leakage flux. For this purpose, it is probably not any more satisfactory than a flux map or a two-dimensional treatment.

The mathematical methods developed are useful for other problems in cylindrical and cartesian coordinates. The problems in two-dimensional cartesian coordinates which can be solved by a modification of the following work can also be solved by a Schwartz transformation if the potential distribution outside the gap is very simple, but for complicated cases, a procedure similar to that given below is to be preferred.

The procedure used is first to separate the variables in Laplace's equation and obtain the following expressions for the scalar potential, one suitable for the region inside the gap, and the other suitable for the outside region.

$$\left. \begin{aligned} \text{I} \\ V &= V_0 \left[z + \sum_{m=1}^{\infty} A_m I_0(\pi m r) \sin(\pi m z) \right] \\ \text{II} \\ V &= V_0 \left[\sum_{m=1}^{\infty} B_m K_0\left(\frac{\pi m r}{z_2}\right) \sin\left(\frac{\pi m z}{z_2}\right) \right. \\ &\quad \left. + \sum_{m=1}^{\infty} C_m K_0(\xi_m r) \sin(\xi_m z) \right] \end{aligned} \right\} \quad (161)$$

where

I
V = scalar magnetic potential in gap
II
V = scalar magnetic potential outside gap
V₀, A_m, B_m, C_m, ξ_m, = are constants
I₀(x) = J₀(ix) ~ Bessel function of imaginary argument
K₀(x) = iH₀⁽¹⁾(ix) ~ Hankel function of imaginary argument

The second series in V does not vanish at the yoke and is used for developing the potential found at this surface. In principle a set of orthogonal functions can be developed for this purpose, but this has not been attempted here.

The coefficients B_m are obtained by making a Fourier analysis of the known potential along the pole and the unknown potential across the gap at $r = r_0$. This unknown potential is expressed in terms of the coefficients A_m and one relation between the A_m and B_m is obtained. This makes the potential continuous at $r = r_0$, $0 < z < 1$. The condition that $\partial V / \partial r$ must be continuous at this boundary gives another set of relations between the sets of coefficients A_m and B_m. The B_m are eliminated between these equations giving the following infinite set of equations in the infinite number of unknowns A_m,

$$A_p + \sum_{m=1}^{\infty} N_{p,m} A_m + M_p = 0 \quad (162)$$

$$p = 1, 2, 3 \dots$$

The A_p form a rapidly converging sequence of numbers in the geometries which are suitable for cyclotrons, and this set of equations is solved to a good approximation, except for a region near the corner of the pole, by using the first three equations in the first three A_m . This, in principle, completes the solution, since the B_m can be obtained from one of the previous equations.

In the following work, the problem, with any potential along the pole, but with the yoke at zero potential, is solved first. This solution is then modified to include any potential on the yoke.

Yoke at Zero Potential

By inspection, and by substituting in Laplace's equation, it can be seen that the following equations satisfy the obvious boundary conditions and the differential equation

$$\begin{aligned} r \leq r_0 \\ 0 \leq z \leq 1 \end{aligned} \quad \begin{aligned} I \\ V = V_0 \left[z + \sum_{m=1}^{\infty} A_m I_0(\pi m r) \sin(\pi m z) \right] \end{aligned} \quad (163)$$

$$\begin{aligned} r \geq r_0 \\ 0 \leq z \leq z_2 = 1/\alpha \end{aligned} \quad \begin{aligned} II \\ V = V_0 \sum_{m=1}^{\infty} B_m K_0(\pi m \alpha r) \sin(\pi m \alpha z) \end{aligned} \quad (164)$$

where $I_0(x) = J_0(ix)$
 $K_0(x) = iH_0^{(1)}(ix)$

i.e., the zero order Bessel and Hankel functions of an imaginary argument. The values of these functions at $r = r_0$ are required, and, therefore, the following notation is used (the V_0 may be suppressed)

$$\begin{aligned} I \\ V_{r_0} = \left[z + \sum_{m=1}^{\infty} \frac{a_m}{\pi m} \sin(\pi m z) \right] = (z + f) \end{aligned} \quad (165)$$

where $a_m = \pi m A_m I_0(\pi m r_0)$, and

$$\begin{aligned} II \\ V_{r_0} = \sum_{m=1}^{\infty} \frac{b_m \beta_m}{\pi m \alpha} \sin(\pi m \alpha z) \end{aligned}$$

where $b_m = \pi m \alpha B_m K_1(\pi m \alpha r_0)$ (166)

$$\beta_m = \frac{K_0(\pi m \alpha r_0)}{K_1(\pi m \alpha r_0)}$$

$$K_1(x) = -H_1^{(1)}(ix)$$

this notation is selected for later convenience. Since these two expressions must be identical when

$$0 \leq z \leq 1$$

and the potential along the pole is given,

$$v = \phi(z) \quad (167)$$

the coefficients b_m may be calculated by a Fourier analysis.

$$\frac{b_n \beta_n}{2\pi n \alpha^2} = \int_0^1 f(z) \sin(\pi n \alpha z) dz \quad \int_0^{1/\alpha} \varphi \sin(\pi n \alpha z) dz \quad (168)$$

where $\varphi = \phi$, $1 \leq z \leq z_2 = 1/\alpha$
 $\varphi = z$, $0 \leq z \leq 1$

The second integral is something that can be immediately evaluated from boundary conditions at the pole surface. It may be written for convenience as

$$\frac{b'_n \beta_n}{2\pi n \alpha^2} \quad (169)$$

The first integral contains the unknown constants a_m and will be written

$$\sum_{n,m}^m \quad (170)$$

and the integration will be carried as far as possible. Using Eq. 165 in the first integral gives

$$\sum_{n,m}^m = \int_0^1 \sin(\pi n \alpha z) \left[\sum_{m=1}^{\infty} \frac{a_m}{\pi m} \sin(\pi m z) \right] dz \quad (171)$$

Interchanging the order of summation and integration, this becomes

$$\sum_{n,m}^m = \sum_{m=1}^{\infty} \frac{a_m}{\pi m} \int_0^1 \sin(\pi n \alpha z) \sin(\pi m z) dz$$

and for any particular term in this series

$$\sum_{n,m} = \frac{a_m}{\pi m} \int_0^1 \sin(\pi n \alpha z) \sin(\pi m z) dz \quad (172)$$

The result is thus

$$\sum_{n,m} = (-1)^m \frac{a_m}{\pi^2} \left(\frac{\sin n\pi\alpha}{n^2\alpha^2 - m^2} \right) \quad (173)$$

Equation 168 becomes

$$b_n = b'_n + \left(\frac{2\pi n \alpha^2}{\beta_n} \right) \sum_{m=1}^{\infty} (-1)^m \frac{\alpha_m}{\pi^2} \left(\frac{\sin n\pi\alpha}{n^2\alpha^2 - m^2} \right) \quad (174)$$

In order to proceed with the solution, the fact that the derivative of V with respect to r is continuous is used. Equations 163 and 164 give

$$\begin{aligned} \frac{I}{H_r} &= -\frac{\partial V}{\partial r} = - \sum \pi m A_m I_1(\pi m r) \sin(\pi m z) \\ \frac{\Pi}{H_r} &= -\frac{\partial V}{\partial r} = \sum \pi m \alpha B_m K_1(\pi m \alpha r) \sin(\pi m \alpha z) \end{aligned} \quad (175)$$

where $I_1(x) = -J_1(ix)$

let

$$\frac{I_1(\pi m r_0)}{I_0(\pi m r_0)} = \gamma_m \quad (176)$$

The two solutions must be continuous at r_0 . This gives, in the notation of Eqs. 165 and 166,

$$\sum_{m=1}^{\infty} a_m \gamma_m \sin(\pi m z) = - \sum_{n=1}^{\infty} b_n \sin(\pi n \alpha z) \quad (177)$$

where $r = r_0$, $0 \leq z \leq 1$

Now solve this equation for a_p by a Fourier analysis.

$$- \frac{a_p \gamma_p}{2} = \sum_{n=1}^{\infty} b_n \int_0^1 \sin(\pi n \alpha z) \sin(\pi p z) dz \quad (178)$$

This integral is the one encountered in Eq. 172 and gives

$$- a_p \gamma_p = \frac{2(-1)^p p}{\pi} \sum_{n=1}^{\infty} b_n \frac{\sin(n\pi\alpha)}{(n^2\alpha^2 - p^2)} \quad (179)$$

Many of the coefficients b_n must be used to represent the solution outside, but only a few a_p are required to represent the solution inside (since the approximate solution is taken care of by $V = z$). For this reason, it is best to eliminate the coefficients b_n in Eq. 179 and get equations in a_p . For this purpose, Eq. 174 may be used.

$$0 = a_p \gamma_p + \frac{2(-1)^p p}{\pi} \sum_{n=1}^{\infty} b_n \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)} + \frac{4(-1)^p p}{\pi^2} \sum_{m=1}^{\infty} (-1)^m a_m \sum_{n=1}^{\infty} \frac{n \alpha^2 \sin^2 \pi n \alpha}{\beta_n (n^2 \alpha^2 - m^2)(n^2 \alpha^2 - p^2)} \quad (180)$$

For convenience, this equation may be written

$$0 = a_p \gamma_p + p M_p + \sum_{m=1}^{\infty} p N_{p,m} a_m \quad (181)$$

where

$$\left. \begin{aligned} M_p &= \left(\frac{2}{\pi}\right) (-1)^p \sum_{n=1}^{\infty} b_n \frac{\sin(\pi n \alpha)}{(n^2 \alpha^2 - p^2)} \\ N_{m,p} &= N_{p,m} = \left(\frac{2}{\pi}\right)^2 (-1)^{p+m} \sum_{n=1}^{\infty} \frac{n \alpha^2 \sin^2(\pi n \alpha)}{\beta_n (n^2 \alpha^2 - m^2)(n^2 \alpha^2 - p^2)} \end{aligned} \right\} \quad (182)$$

It is clear that Eq. 181 represents an infinite number of equations, and that there are infinitely many unknowns, a_p , in each. The problem is perfectly definite since the coefficients M_p and $N_{p,m}$ may be determined for each equation. It is fortunate that a limited number of these equations, each with a limited number of terms, gives a quite accurate solution to the problem which may be made as accurate as necessary by including more equations and more unknowns.

Consider now a particular form for $\varphi(z)$ (see Eq. 168) so that the M_p may be determined explicitly. The potential rises linearly over a thin coil, provided the surface density of current is constant. Therefore

$$\begin{aligned}
\varphi &= z, \quad 0 \leq z \leq 1 \\
\varphi &= 1, \quad 1 \leq z \leq z_1 \\
\varphi &= \frac{z_2 - z}{z_2 - z_1}, \quad z_1 \leq z \leq z_2
\end{aligned} \tag{183}$$

From Eqs. 168, 169, and 183,

$$b'_n = \frac{2\pi n \alpha^2}{\beta_n} \left[\int_0^1 z \sin(\pi n \alpha z) dz + \int_1^{z_1} \sin(n\pi \alpha z) dz + \int_{z_1}^{z_2} \left(\frac{z_2 - z}{z_2 - z_1} \right) \sin(n\pi \alpha z) dz \right] \tag{184}$$

Performing the indicated integrations gives

$$b'_n = \frac{2}{\pi \beta_n n} \left[\sin(\pi n \alpha) + \left(\frac{1}{z_2 - z_1} \right) \sin(z_1 \pi n \alpha) \right] \tag{185}$$

Putting this result in Eq. 182 gives

$$M'_p = \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{n=1}^{\infty} \frac{1}{\beta_n} \frac{\sin^2(\pi n \alpha)}{n(n^2 \alpha^2 - p^2)} \tag{186}$$

$$M''_p = \left(\frac{2}{\pi} \right)^2 \frac{(-1)^p}{z_2 - z_1} \sum_{n=1}^{\infty} \frac{\sin(\pi n \alpha) \sin(z_1 \pi n \alpha)}{\beta_n n(n^2 \alpha^2 - p^2)} \tag{187}$$

where $M_p = M'_p + M''_p$.

Methods for summing these slowly convergent series are required. The various series to be considered are given in Eqs. 182, 185, and 187. The Poisson summation formula may be written in a way which applies to this type of series.

The Poisson summation formula is

$$\sum_{n=-\infty}^{\infty} \phi(n) = \sum_{h=-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t) e^{-2\pi i h t} dt \tag{188}$$

If $\phi(n)$ is an even function and $\phi(0) = 0$ then

$$\sum_{n=-\infty}^{\infty} \phi(n) = 2 \sum_{n=1}^{\infty} \phi(n) \tag{189}$$

The integral in Eq. 188 may be written

$$\int_0^{\infty} \phi(t) e^{-2\pi i h t} dt + \int_{-\infty}^0 \phi(s) e^{-2\pi i h s} ds \tag{190}$$

Now let $s = -t$; then $ds = -dt$ and $\phi(s) = \phi(-t) = \phi(t)$, and Eq. 190 becomes

$$\int_0^{\infty} \phi(t) e^{-2\pi i h t} dt - \int_{\infty}^0 \phi(t) e^{2\pi i h t} dt$$

Changing the limits on the last integral

$$\int_0^{\infty} \phi(t) (e^{2\pi i h t} + e^{-2\pi i h t}) dt = 2 \int_0^{\infty} \phi(t) \cos(2\pi h t) dt \tag{191}$$

This result is now independent of whether $\phi(n)$ is an even function or not.

The summation formula becomes from Eqs. 189 and 191

$$\sum_{n=1}^{\infty} \phi(n) = \sum_{h=-\infty}^{\infty} \int_0^{\infty} \phi(t) \cos(2\pi ht) dt \quad (192)$$

provided

$$\phi(0) = 0$$

It may be more convenient to use the formula in the form

$$\sum_{n=1}^{\infty} \phi(n) = \int_0^{\infty} \phi(t) dt + 2 \sum_{h=1}^{\infty} \int_0^{\infty} \phi(t) \cos(2\pi ht) dt \quad (193)$$

If the condition $\phi(0) = 0$ is not satisfied, then the left-hand side of Eq. 193 may be replaced by

$$\frac{1}{2} \phi(0) + \sum_{n=1}^{\infty} \phi(n) \quad (194)$$

The series to be summed may now be written in the form

$$N_{p,m} = \left(\frac{2}{\pi}\right)^2 (-1)^{p+m} \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{1}{\beta(x)} \frac{x\alpha^2 \sin^2(\pi\alpha x) \cos(2\pi hx)}{(x^2\alpha^2 - m^2)(x^2\alpha^2 - p^2)} dx \quad (195)$$

$$M'_p = \left(\frac{2}{\pi}\right)^2 (-1)^p \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{1}{\beta(x)} \frac{\sin^2(\pi\alpha x) \cos(2\pi hx)}{x(x^2\alpha^2 - p^2)} dx \quad (196)$$

$$M''_p = \left(\frac{2}{\pi}\right)^2 \frac{(-1)^p}{(z_2 - z_1)} \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{1}{\beta(x)} \frac{\sin(\pi\alpha x) \sin(\pi\alpha z_1 x) \cos(2\pi hx)}{x(x^2\alpha^2 - p^2)} dx \quad (197)$$

The factor $1/\beta(x)$ in these integrals makes them quite complicated. Since the $1/\beta_n$ are close to one for all n in the problems considered here and rapidly approach one as n increases, the series may be summed by setting $\beta_n = 1$ and then correcting a few terms in each series. The following expressions are obtained when the $\beta_n = 1$,

$$\bar{N}_{p,m} = \left(\frac{2}{\pi}\right)^2 (-1)^{p+m} \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{x\alpha^2 \sin^2(\pi\alpha x) \cos(2\pi hx)}{(x^2\alpha^2 - m^2)(x^2\alpha^2 - p^2)} dx \quad (198)$$

$$\bar{M}'_p = \left(\frac{2}{\pi}\right)^2 (-1)^p \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{\sin^2(\pi\alpha x) \cos(2\pi hx)}{x(x^2\alpha^2 - p^2)} dx \quad (199)$$

$$\bar{M}''_p = \frac{1}{(z_2 - z_1)} \left(\frac{2}{\pi}\right)^2 (-1)^p \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{\sin(\pi\alpha x) \sin(\pi\alpha z_1 x) \cos(2\pi hx)}{x(x^2\alpha^2 - p^2)} dx \quad (200)$$

The general form of the integral to be considered is

$$I = a^2 \int_0^{\infty} \frac{y \sin(\pi\alpha y) \sin(\pi\beta y) \cos(2\pi hy)}{(y^2\alpha^2 - m^2)(y^2\alpha^2 - p^2)} dy \quad (201)$$

Putting $x = \alpha y$ the integral takes the form,

$$I = \int_0^{\infty} \frac{x \sin(\pi x) \sin(\pi\beta x) \cos\left[\frac{2\pi(h/\alpha)x}{\alpha}\right]}{(x^2 - m^2)(x^2 - p^2)} dx \quad (202)$$

Denoting $h/\alpha = \gamma$, the problem is reduced to the evaluation of the integral

$$I = \int_0^\infty \frac{x \sin(\pi x) \sin(\pi \beta x) \cos(2\pi \gamma x)}{(x^2 - p^2)(x^2 - q^2)} dx \quad (203)$$

The integral may be broken up into a sum of simple integrals by partial fractions,

$$\frac{x}{(x^2 - p^2)(x^2 - q^2)} = \frac{1}{2(p^2 - q^2)} \left(\frac{1}{x - p} + \frac{1}{x + p} - \frac{1}{x - q} - \frac{1}{x + q} \right) \quad (204)$$

and

$$\sin(\pi x) \sin(\pi \beta x) \cos(2\pi \gamma x) = \frac{1}{4} \left[\cos(\beta - 1 - 2\gamma) \pi x + \cos(\beta - 1 + 2\gamma) \pi x \right. \\ \left. - \cos(\beta + 1 - 2\gamma) \pi x - \cos(\beta + 1 + 2\gamma) \pi x \right] \quad (205)$$

Of the 16 resulting integrals, choose a typical one,

$$\int_0^\infty \frac{\cos(\eta \pi x)}{x + m} dx = \int_0^\infty \frac{\cos(\eta \pi x) d[\eta \pi(x + m)]}{\eta \pi(x + m)} \quad (206)$$

Expand

$$\cos[\eta \pi(x + m) - \eta \pi m] = \cos \eta \pi m \cos \eta \pi(x + m) + \sin \eta \pi m \sin \eta \pi(x + m)$$

let $y = \eta \pi(x + m)$, and the integral becomes

$$\cos(\eta \pi m) \int_{\eta \pi m}^\infty \frac{\cos y dy}{y} + \sin \eta \pi m \int_{\eta \pi m}^\infty \frac{\sin y dy}{y} \quad (207)$$

and these are defined integrals

$$- \cos \eta \pi m \text{Ci } \eta \pi m - \sin \eta \pi m \text{Si } \pi m \eta \quad (208)$$

or

$$- \cos \eta \pi m \text{Ci } \eta \pi m - \sin \eta \pi m \text{Si } \pi m \eta + \frac{\pi}{2} \sin \eta \pi m \quad (209)$$

where Ci is the cosine integral and Si the sine integral.

Since the whole integral will contain 32 terms, it will be wise to do a pair of terms.

$$\frac{1}{8(p^2 - q^2)} \int_0^\infty \left[\left(\frac{\cos \eta \pi x}{x + m} \right) + \left(\frac{\cos \eta \pi x}{x - m} \right) \right] dx \quad (210)$$

$$= \frac{1}{8(p^2 - q^2)} \left[- \cos(\eta \pi m) \text{Ci } (\eta \pi m) - \sin(\eta \pi m) \text{Si } (\eta \pi m) + \frac{\pi}{2} \sin(\eta \pi m) \right. \\ \left. - \cos(-\eta \pi m) \text{Ci } (\eta \pi m) - \sin(-\eta \pi m) \text{Si } (-\eta \pi m) + \frac{\pi}{2} \sin(-\eta \pi m) \right] \quad (211)$$

$$= \frac{-1}{4(p^2 - q^2)} \left[\cos(\eta \pi m) \text{Ci } (\eta \pi m) + \sin(\eta \pi m) \text{Si } (\eta \pi m) \right] \quad (212)$$

Since there are eight terms of the above form, denote this bracket by $+Y(\eta \pi m)$. Thus the above integral is

$$= \frac{-1}{4(p^2 - q^2)} Y(\eta\pi m) \quad (213)$$

Note that $Y(-x) = Y(x)$. Thus the integral, I , may be written, in general, as

$$I = \frac{-1}{4(p^2 - q^2)} [Y(\beta - 1 - 2\gamma)\pi p - Y(\beta - 1 - 2\gamma)\pi q \\ + Y(\beta - 1 + 2\gamma)\pi p - Y(\beta - 1 + 2\gamma)\pi p \\ - Y(\beta + 1 - 2\gamma)\pi p + Y(\beta + 1 - 2\gamma)\pi q \\ - Y(\beta + 1 + 2\gamma)\pi p + Y(\beta + 1 + 2\gamma)\pi q] \quad (214)$$

Expanding this expression gives, provided that p and q are integers,

$$I = \frac{-1}{4(p^2 - q^2)} \{ \cos(\beta + 1 - 2\gamma)\pi p [Ci(\beta - 1 - 2\gamma)\pi p - Ci(\beta + 1 - 2\gamma)\pi p] \\ + \sin(\beta + 1 - 2\gamma)\pi p [Si(\beta - 1 - 2\gamma)\pi p - Si(\beta + 1 - 2\gamma)\pi p] \\ + \cos(\beta + 1 + 2\gamma)\pi p [Ci(\beta - 1 + 2\gamma)\pi p - Ci(\beta + 1 + 2\gamma)\pi p] \\ + \sin(\beta + 1 + 2\gamma)\pi p [Si(\beta - 1 + 2\gamma)\pi p - Si(\beta + 1 + 2\gamma)\pi p] \\ - \cos(\beta + 1 - 2\gamma)\pi q [Ci(\beta - 1 - 2\gamma)\pi q - Ci(\beta + 1 - 2\gamma)\pi q] \\ - \sin(\beta + 1 - 2\gamma)\pi q [Si(\beta - 1 - 2\gamma)\pi q - Si(\beta + 1 - 2\gamma)\pi q] \\ - \cos(\beta + 1 + 2\gamma)\pi q [Ci(\beta - 1 + 2\gamma)\pi q - Ci(\beta + 1 + 2\gamma)\pi q] \\ - \sin(\beta + 1 + 2\gamma)\pi q [Si(\beta - 1 + 2\gamma)\pi q - Si(\beta + 1 + 2\gamma)\pi q] \} \quad (215)$$

If β , 2γ , p and q are integers, the sine terms drop out and the integral becomes

$$I = \frac{-1}{4(p^2 - q^2)} \{ \cos(\beta + 1 + 2\gamma)\pi p [Ci(\beta - 1 - 2\gamma)\pi p - Ci(\beta + 1 - 2\gamma)\pi p] \\ + Ci(\beta - 1 + 2\gamma)\pi p - Ci(\beta + 1 + 2\gamma)\pi p \\ - \cos(\beta + 1 + 2\gamma)\pi q [Ci(\beta - 1 - 2\gamma)\pi q - Ci(\beta + 1 - 2\gamma)\pi q] \\ + Ci(\beta - 1 + 2\gamma)\pi q - Ci(\beta + 1 + 2\gamma)\pi q \} \quad (216)$$

The cosine terms in the above expression are either $+1$ or -1 , depending on the argument. If one wishes to evaluate these expressions when p , q , γ , or $\beta - 1$ goes to zero, it is, in general, necessary to take the limit as the appropriate variable or variables vanish.

There exist semiconvergent series for Si and Ci which are convenient for large arguments. These are $x \gg \pi$,

$$Ci(x) = \frac{\sin x}{x} \left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \frac{6!}{x^6} + \dots \right) \\ - \frac{\cos x}{x} \left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \frac{7!}{x^7} + \dots \right) \quad (217)$$

$$Si(x) = \frac{\cos x}{x} \left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \frac{6!}{x^6} + \dots \right) \\ - \frac{\sin x}{x} \left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \frac{7!}{x^7} + \dots \right) \quad (218)$$

For convenience, denote these by the following expressions:

$$Ci(x) = \frac{\sin(x)}{x} U(x) - \frac{\cos(x)}{x} V(x) \quad (219)$$

$$Si(x) = -\frac{\cos(x)}{x} U(x) - \frac{\sin(x)}{x} V(x) \quad (220)$$

These expressions may be put in the equation for the function Y (see Eq. 213).

$$Y(\eta\pi m) = \cos(\eta\pi m) \text{Ci}(\eta\pi m) + \sin(\eta\pi m) \text{Si}(\eta\pi m)$$

giving

$$Y(x) = \frac{\cos(x) \sin(x) U(x)}{x} - \frac{\cos^2(x) V(x)}{x} - \frac{\sin(x) \cos(x) U(x)}{x} - \frac{\sin^2(x) V(x)}{x} + \frac{\pi}{2} \sin x$$

or

$$Y(x) = -\frac{V(x)}{x} + \frac{\pi}{2} \sin x = -\frac{1}{x^2} \left(1 - \frac{3!}{x^2} + \frac{5!}{x^4} - \dots \right) + \frac{\pi}{2} \sin x \quad (221)$$

This may be used in Eq. 214, if $x \gg \pi$. If p and q are integers, the expression can be greatly simplified. Considering a term similar to the first and the fifth terms of Eq. 214 gives

$$Y(x) - Y(x + 2n\pi) = -\frac{V(x)}{x} + \frac{\pi}{2} \sin x + \frac{V(x + 2n\pi)}{x + 2n\pi} - \frac{\pi}{2} \sin x = \frac{V(x + 2n\pi)}{x + 2n\pi} - \frac{V(x)}{x} \quad (222)$$

Thus, Eq. 203 may be expressed to this approximation by $x \gg \pi$, p and q integers, and taking only the first term in the V series (which is equivalent to taking three terms in the Si and Ci series).

$$I = \frac{2\beta}{p^2 q^2 \pi^2} \frac{[48\gamma^4 - 8\gamma^2(\beta^2 + 1) - (\beta^2 - 1)^2]}{[16\gamma^4 - 8\gamma^2(\beta^2 + 1) + (\beta^2 - 1)^2]^2} \quad (223)$$

Consider now $\bar{N}_{p,p}$. From Eq. 198 this may be written

$$\bar{N}_{p,p} = \left(\frac{2}{\pi}\right)^2 \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{x\alpha^2 \sin^2(\pi\alpha x) \cos(2\pi hx)}{(x^2\alpha^2 - p^2)^2} dx \quad (224)$$

This is in the form of Eq. 203, if $\gamma = h/\alpha$, $\beta = 1$, and $q = p$.

The integrals are therefore given by Eq. 214. Substituting $q = p$ gives zero; therefore, let q approach p. Therefore, take the derivative of the numerator of Eq. 214 with respect to q, divide by the derivative of the denominator with respect to q, and let q approach p. The derivative of the numerator is

$$\left\{ \begin{aligned} & -(\beta - 1 - 2\gamma)\pi [-\sin(\beta - 1 - 2\gamma)\pi q \text{Ci}(\beta - 1 - 2\gamma)\pi q + \cos(\beta - 1 - 2\gamma)\pi q \text{Si}(\beta - 1 - 2\gamma)\pi q] - \frac{1}{q} \\ & -(\beta - 1 + 2\gamma)\pi [-\sin(\beta - 1 + 2\gamma)\pi q \text{Ci}(\beta - 1 + 2\gamma)\pi q + \cos(\beta - 1 + 2\gamma)\pi q \text{Si}(\beta - 1 + 2\gamma)\pi q] - \frac{1}{q} \\ & +(\beta + 1 - 2\gamma)\pi [-\sin(\beta + 1 - 2\gamma)\pi q \text{Ci}(\beta + 1 - 2\gamma)\pi q + \cos(\beta + 1 - 2\gamma)\pi q \text{Si}(\beta + 1 - 2\gamma)\pi q] + \frac{1}{q} \\ & +(\beta + 1 + 2\gamma)\pi [-\sin(\beta + 1 + 2\gamma)\pi q \text{Ci}(\beta + 1 + 2\gamma)\pi q + \cos(\beta + 1 + 2\gamma)\pi q \text{Si}(\beta + 1 + 2\gamma)\pi q] + \frac{1}{q} \end{aligned} \right\} \quad (225)$$

Now let $\beta = 1$.

$$\begin{aligned} 4\pi\gamma (\sin 2\pi\gamma q \text{Ci } 2\pi\gamma q - \cos 2\pi\gamma q \text{Si } 2\pi\gamma q) \\ + 2(\gamma - 1)\pi [-\sin 2(\gamma - 1)\pi q \text{Ci } 2(\gamma - 1)\pi q + \cos 2(\gamma - 1)\pi q \text{Si } 2(\gamma - 1)\pi q] \\ + 2(\gamma + 1)\pi [-\sin 2(\gamma + 1)\pi q \text{Ci } 2(\gamma + 1)\pi q + \cos 2(\gamma + 1)\pi q \text{Si } 2(\gamma + 1)\pi q] \end{aligned} \quad (226)$$

The derivative of the denominator is

$$\frac{d[-4(p^2 - q^2)]}{dq} = 8q \quad (227)$$

Thus the value of I when $\beta = 1$ and $q \rightarrow p$ is

$$I = \frac{\pi}{4p} \left\{ 2\gamma (\sin 2\pi\gamma p \text{ Ci } 2\pi\gamma p - \cos 2\pi\gamma p \text{ Si } 2\pi\gamma p) \right. \\ \left. - \sin 2\pi\gamma p [(\gamma - 1) \text{ Ci } 2(\gamma - 1)\pi p + (\gamma + 1) \text{ Ci } 2(\gamma + 1)\pi p] \right. \\ \left. + \cos 2\pi\gamma p [(\gamma - 1) \text{ Si } 2(\gamma - 1)\pi p + (\gamma + 1) \text{ Si } 2(\gamma + 1)\pi p] \right\} \quad (228)$$

The value of $\bar{N}_{p,p}$ from Eqs. 224 and 228 is

$$p \bar{N}_{p,p} = + \frac{2}{\pi} \text{Si } (2\pi p) + \frac{2}{\pi} \sum_{h=1}^{\infty} \frac{2h}{a} \left\{ \sin \frac{2h}{a} \pi p \text{ Ci } \left(2\pi \frac{h}{a} p \right) - \cos \left(2\pi \frac{h}{a} p \right) \text{Si } \left(2\pi \frac{h}{a} p \right) \right. \\ \left. - \sin 2\pi \frac{h}{a} p \left[\left(\frac{h}{a} - 1 \right) \text{Ci } 2 \left(\frac{h}{a} - 1 \right) \pi p + \left(\frac{h}{a} + 1 \right) \text{Ci } 2 \left(\frac{h}{a} + 1 \right) \pi p \right] \right. \\ \left. + \cos 2\pi \frac{h}{a} p \left[\left(\frac{h}{a} - 1 \right) \text{Si } 2 \left(\frac{h}{a} - 1 \right) \pi p + \left(\frac{h}{a} + 1 \right) \text{Si } 2 \left(\frac{h}{a} + 1 \right) \pi p \right] \right\} \quad (229)$$

This equation can be expanded in terms of the semiconvergent series for Si and Ci. This gives a correction term which is vanishingly small whenever the semiconvergent series is valid, and this evaluation is, therefore, of no use. The lower limit of the argument for which the semiconvergent series is useful may be set at 4π . Thus, no correction (<0.1 per cent) will be encountered when

$$m \left(\frac{h}{a} - 1 \right) \geq 2$$

For all values of α up to $1/3$

$$p \bar{N}_{p,p} = + \frac{2}{\pi} \text{Si } 2\pi p \quad (230)$$

Consider now $\bar{N}_{p,m}$, given in Eq. 199, when $p \neq m$. The integrals are in the form of Eq. 203, if $\beta = 1$ and $\gamma = h/\alpha$. Thus

$$\bar{N}_{p,m} = \frac{(2)^2}{\pi} (-1)^{p+m} \sum_{h=-\infty}^{\infty} I \quad (231)$$

$$\beta = 1 \quad \gamma = \frac{h}{\alpha}$$

Putting $\beta = 1$ and $\gamma = h/\alpha$ in Eq. 215 gives

$$\bar{N}_{p,q} = - \frac{(-1)^{p+q}}{\pi^2(p^2 - q^2)} \left\{ \cos 2\pi \frac{h}{a} p \left[2 \text{Ci } 2\pi \frac{h}{a} p - \text{Ci } 2 \left(\frac{h}{a} - 1 \right) \pi p - \text{Ci } 2 \left(\frac{h}{a} + 1 \right) \pi p \right] \right. \\ \left. + \sin 2\pi \frac{h}{a} p \left[2 \text{Si } 2\pi \frac{h}{a} p - \text{Si } 2 \left(\frac{h}{a} - 1 \right) \pi p - \text{Si } 2 \left(\frac{h}{a} + 1 \right) \pi p \right] \right. \\ \left. - \cos 2\pi \frac{h}{a} q \left[2 \text{Ci } 2\pi \frac{h}{a} q - \text{Ci } 2 \left(\frac{h}{a} - 1 \right) \pi q - \text{Ci } 2 \left(\frac{h}{a} + 1 \right) \pi q \right] \right. \\ \left. - \sin 2\pi \frac{h}{a} q \left[2 \text{Si } 2\pi \frac{h}{a} q - \text{Si } 2 \left(\frac{h}{a} - 1 \right) \pi q - \text{Si } 2 \left(\frac{h}{a} + 1 \right) \pi q \right] \right\} \quad (232)$$

In order to determine the form of the term where $h = 0$, let $h = \epsilon$ and let $\epsilon \rightarrow 0$. The resulting equation for the brackets above is

$$\lim_{\epsilon \rightarrow 0} \left(2 \operatorname{Ci} \frac{2\pi p}{\alpha} \epsilon - 2 \operatorname{Ci} \frac{2\pi q}{\alpha} \epsilon - 2 \operatorname{Ci} 2\pi p + 2 \operatorname{Ci} 2\pi q \right) \quad (233)$$

Using the formula

$$\operatorname{Ci} x \rightarrow C + \log(x) \quad (234)$$

if $x \rightarrow 0$, for the cosine integral of a small argument, Eq. 233 becomes

$$2 \left[\log \frac{p}{q} - \operatorname{Ci} 2\pi p + \operatorname{Ci} 2\pi q \right] \quad (235)$$

As in the previous terms, $\bar{N}_{m,m}$, the series $h \neq 0$ gives a negligible contribution when α is not greater than $1/3$ (<1 per cent). Thus

$$\bar{N}_{p,q} = \frac{2(-1)^{p+q}}{\pi^2(q^2 - p^2)} \left[\log \frac{p}{q} - \operatorname{Ci} 2\pi p + \operatorname{Ci} 2\pi q \right] \quad (236)$$

When $p \geq 2$ and $q \geq 2$, the semiconvergent series (Eq. 219) may be used for the cosine integrals,

$$\bar{N}_{p,q} = \frac{2(-1)^{p+q}}{\pi^2(q^2 - p^2)} \log \left(\frac{p}{q} \right) + \frac{(-1)^{p+q}}{2\pi^4 p^2 q^2} \quad (237)$$

And here the last term may be dropped in working to 1 per cent in this cross-term coefficient.

Consider next the term $\bar{M}''_p$. This is given by

$$\bar{M}''_p = \frac{1}{z_2 - z_1} \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{h=-\infty}^{\infty} I \quad (238)$$

$$\gamma = \frac{h}{\alpha} \quad q = 0 \quad \beta = z_1$$

The value of I may be deduced from Eq. 215 by letting $q = \epsilon \rightarrow 0$, using Eq. 234,

$$\begin{aligned} I(\gamma = \frac{h}{\alpha} \quad q = 0) = \frac{-1}{4p^2} & \left\{ \cos \left(\beta + 1 - \frac{2h}{\alpha} \right) \pi p \left[\operatorname{Ci} \left(\beta - 1 - \frac{2h}{\alpha} \right) \pi p - \operatorname{Ci} \left(\beta + 1 - \frac{2h}{\alpha} \right) \pi p \right] \right. \\ & + \sin \left(\beta + 1 - \frac{2h}{\alpha} \right) \pi p \left[\operatorname{Si} \left(\beta - 1 - \frac{2h}{\alpha} \right) \pi p - \operatorname{Si} \left(\beta + 1 - \frac{2h}{\alpha} \right) \pi p \right] \\ & + \cos \left(\beta + 1 + \frac{2h}{\alpha} \right) \pi p \left[\operatorname{Ci} \left(\beta - 1 + \frac{2h}{\alpha} \right) \pi p - \operatorname{Ci} \left(\beta + 1 + \frac{2h}{\alpha} \right) \pi p \right] \\ & + \sin \left(\beta + 1 + \frac{2h}{\alpha} \right) \pi p \left[\operatorname{Si} \left(\beta - 1 + \frac{2h}{\alpha} \right) \pi p - \operatorname{Si} \left(\beta + 1 + \frac{2h}{\alpha} \right) \pi p \right] \\ & \left. + \log \frac{4 \left(\frac{h}{\alpha} \right)^2 + (\beta + 1)^2}{4 \left(\frac{h}{\alpha} \right)^2 + (\beta - 1)^2} \right\} \quad (239) \end{aligned}$$

The value of I when $h = 0$ is thus

$$\begin{aligned} I(\gamma = 0 \quad q = 0) = \frac{-1}{2p^2} & \left\{ \cos(\beta + 1) \pi p \left[\operatorname{Ci}(\beta - 1) \pi p - \operatorname{Ci}(\beta + 1) \pi p \right] \right. \\ & + \sin(\beta + 1) \pi p \left[\operatorname{Si}(\beta - 1) \pi p - \operatorname{Si}(\beta + 1) \pi p \right] + \log \left(\frac{\beta + 1}{\beta - 1} \right) \left. \right\} \quad (240) \end{aligned}$$

Expansion of the Ci and Si in the semiconvergent series (valid when $\sim 2h/\alpha \geq 4 + \beta$) indicates that the zero term is sufficient (~ 1 per cent). Therefore,

$${}_p \bar{M}'_p = \frac{-1}{(z_2 - z_1)} \left(\frac{2}{\pi}\right)^2 (-1)^p \frac{1}{2p} \left\{ \cos(\beta + 1)\pi p [Ci(\beta - 1)\pi p - Ci(\beta + 1)\pi p] \right. \\ \left. + \sin(\beta + 1)\pi p [Si(\beta - 1)\pi p - Si(\beta + 1)\pi p] + \log\left(\frac{\beta + 1}{\beta - 1}\right) \right\} + W \quad (241)$$

$$W = \sum_{h=1}^{\infty} \log \left[\frac{4\left(\frac{h}{\alpha}\right)^2 - (\beta + 1)^2}{4\left(\frac{h}{\alpha}\right)^2 - (\beta - 1)^2} \right]$$

The coefficient $\bar{M}'_p$ may be simply derived from the above by changing the factor outside and letting $\beta = 1 + \epsilon$ and $\epsilon \rightarrow 0$,

$${}_p \bar{M}'_p = -\left(\frac{2}{\pi}\right)^2 (-1)^p \frac{1}{2p} \left[C + \log(2\pi p) - Ci(2\pi p) + \sum_{h=1}^{\infty} \log\left(\frac{1 - \alpha^2}{h^2}\right) \right] \quad (242)$$

where C = Eulers constant.

This completes the solution when the potential along the yoke is zero. It is seen that an approximate solution is not difficult to obtain since the leading terms in the uncorrected coefficients ($\bar{N}_{m,n}$, $\bar{N}_{m,m}$, $\bar{M}_p$) are simple tabulated functions.

Yoke with Known Potential $V = V(r)$

By a procedure similar to that used above, it is possible to analyze any potential along the yoke. In this case, a more general potential function is required outside the gap.

$$\frac{\Pi}{V} = \sum_{m=1}^{\infty} B_m K_0\left(\frac{\pi m r}{z_2}\right) \sin\left(\frac{\pi m z}{z_2}\right) + \sum_s \mu_s K_0\left(\frac{\pi \eta_s r}{z_2}\right) \sin\left(\frac{\pi \eta_s z}{z_2}\right) \quad (243)$$

The values of the B_m are determined by a Fourier analysis of the potential at $r = r_0$.

$$\sum_{m=1}^{\infty} B_m K_0\left(\frac{\pi m r_0}{z_2}\right) \sin\left(\frac{\pi m z}{z_2}\right) = - \sum_s \mu_s K_0\left(\frac{\pi \eta_s r_0}{z_2}\right) \sin\left(\frac{\pi \eta_s z}{z_2}\right) \quad (244)$$

(See Eqs. 165 and 168.) Using the previous notation

$$\frac{b_n \beta_n}{2\pi n \alpha^2} = - \sum_s \frac{\nu_s \sigma_s}{\alpha \pi \eta_s} \int_0^{z_2} \sin(\pi \eta_s \alpha z) \sin(\pi n \alpha z) dz + \int_0^{z_2} \phi \sin(\pi n \alpha z) dz + \int_0^1 f \sin(\pi n \alpha z) dz \quad (245)$$

where as before ($\alpha = 1/z_2$) and

$$\left[\nu_s = \alpha \pi \eta_s \mu_s K_1(\pi \eta_s \alpha r_0) \right] \quad \left[\sigma_s = \frac{K_0(\pi \eta_s \alpha r_0)}{K_1(\pi \eta_s \alpha r_0)} \right]$$

These expressions are readily integrated to give

$$b_n = b'_n + \frac{2}{\pi} \frac{n \alpha^2}{\beta_n} \sum_{m=1}^{\infty} a_m (-1)^m \left(\frac{\sin n \pi \alpha}{n^2 \alpha^2 - m^2} \right) - \frac{2}{\pi} \sum_s \nu_s \frac{\sigma_s}{\beta_n} \frac{n^2 (-1)^n}{\eta_s} \left(\frac{\sin \pi \eta_s}{\eta_s^2 - n^2} \right) \quad (246)$$

where b'_n are the coefficients of the potential function which exists along the pole plus the first term of the gap series, as before.

As before, it is necessary to use the continuity of H_r at $r = r_0$ ($0 \leq z \leq 1$) to get a second set of equations relating the a_n and b_m .

$$\begin{aligned} \text{I} \\ H_r = -\frac{\partial V}{\partial r} = -\sum_{m=1}^{\infty} \pi m A_m I_1(\pi m r) \sin(\pi m z) \\ \text{II} \\ H_r = -\frac{\partial V}{\partial r} = \sum_{m=1}^{\infty} \pi m \alpha B_m K_1(\pi m \alpha r) \sin(\pi m \alpha z) + \sum_S \pi \eta_S \alpha \mu_S K_1(\pi \eta_S \alpha r) \sin(\pi \eta_S \alpha z) \end{aligned} \quad (247)$$

Equating these values at $r = r_0$ and using the new notation gives

$$-\sum_{m=1}^{\infty} a_m \gamma_m \sin(\pi m z) = \sum_{n=1}^{\infty} b_n \sin(\pi n \alpha z) + \sum_S \nu_S \sin(\pi \eta_S \alpha z) \quad (248)$$

Now solve for a_p by a Fourier analysis.

$$-\frac{a_p \gamma_p}{2} = \sum_{n=1}^{\infty} b_n \int_0^1 \sin(\pi n \alpha z) \sin(\pi p z) dz + \sum_S \nu_S \int_0^1 \sin(\pi \eta_S \alpha z) \sin(\pi p z) dz \quad (249)$$

Integrating gives

$$-a_p \gamma_p = \sum_n b_n \frac{2}{\pi} p(-1)^p \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)} + \sum_S \nu_S \frac{2}{\pi} p(-1)^p \frac{\sin \pi \eta_S \alpha}{(\eta_S^2 \alpha^2 - p^2)} \quad (250)$$

Putting the previous result for b_n into this set gives a set of equations for the a_p .

$$\begin{aligned} 0 = a_p \gamma_p + \frac{2}{\pi} p(-1)^p \sum_n b'_n \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)} \\ + \left(\frac{2}{\pi}\right)^2 p(-1)^p \sum_{n=1}^{\infty} \frac{n \alpha^2}{\beta_n} \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)} \sum_{m=1}^{\infty} a_m (-1)^m \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - m^2)} \\ - \left(\frac{2}{\pi}\right)^2 p(-1)^p \sum_{n=1}^{\infty} \frac{n^2 (-1)^n}{\beta_n} \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)} \sum_S \nu_S \frac{\sigma_S}{\eta_S} \frac{\sin \pi \eta_S}{(\eta_S^2 - n^2)} \\ + \frac{2}{\pi} p(-1)^p \sum_S \nu_S \frac{\sin \pi \eta_S \alpha}{(\eta_S^2 \alpha^2 - p^2)} \end{aligned} \quad (251)$$

Which is written, for convenience,

$$0 = a_p \gamma_p + p M_p + \sum_{m=1}^{\infty} p N_{p,m} a_m + p \sum_S (Q_{p,S} + R_{p,S}) \quad (252)$$

where

$$M_p = \frac{2}{\pi} (-1)^p \sum_n b'_n \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)}$$

$$\begin{aligned}
\frac{2}{\pi} (-1)^p \nu_s \frac{\sin \pi \eta_s \alpha}{(\eta_s^2 \alpha^2 - p^2)} &= Q_{p,s} \\
R_{p,s} &= - \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{n=1}^{\infty} \left(\frac{n^2 (-1)^n}{\beta_n} \right) \left(\frac{\nu_s \sigma_s}{\eta_s} \right) \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)} \frac{\sin \pi \eta_s}{(\eta_s^2 - n^2)} \\
N_{p,m} = N_{m,p} &= \left(\frac{2}{\pi} \right)^2 (-1)^{(p+m)} \sum_{n=1}^{\infty} \frac{n \alpha^2}{\beta_n} \frac{\sin^2 \pi n \alpha}{(n^2 \alpha^2 - p^2)(n^2 \alpha^2 - m^2)}
\end{aligned} \tag{253}$$

The expressions for M_p and $N_{p,m}$ are formally what they were before, the M_p depending only on the potential distribution ϕ .

The b'_n may be determined for a particular function ϕ . Let

$$\begin{aligned}
\varphi &= z & r &= r_0 & 0 \leq z \leq 1 \\
\varphi &= 1 & r &= r_0 & 1 \leq z \leq z_2
\end{aligned} \tag{254}$$

That is, the potential over the pole is V_0 . This is a special case of the previous problem, Eq. 183, where $z_1 = z_2$. In order to evaluate Eq. 185 for $z_1 = z_2$, take the ratio of the derivative of the numerator to the derivative of the denominator,

$$b'_n = \frac{2}{\pi \beta_n n} [\sin(n\pi\alpha) - (-1)^n \pi n \alpha] \tag{255}$$

Clearly, the M'_p are the same as in Eq. 186. The M''_p , however, become

$$M''_p = - \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{n=1}^{\infty} \frac{(-1)^n \pi \alpha \sin n\pi\alpha}{\beta_n (n^2 \alpha^2 - p^2)} \tag{256}$$

which can be written

$$M''_p = - \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{n=1}^{\infty} \frac{\pi \alpha \sin n\pi(\alpha + 1)}{\beta_n (n^2 \alpha^2 - p^2)} \tag{257}$$

By the summation formula, this becomes

$$M''_p = - \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{\pi \alpha \sin \pi(\alpha + 1)x \cos 2\pi h x}{\beta(x)(x^2 \alpha^2 - p^2)} dx \tag{258}$$

and the approximate solution is $y = \alpha x$ and $\beta \sim 1$.

$$\bar{M}''_p = - \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{\pi \sin \pi \left(\frac{\alpha + 1}{\alpha} \right) y \cos 2\pi \frac{h}{\alpha} y}{(y^2 - p^2)} dy \tag{259}$$

The $Q_{p,s}$ require no summation. The $R_{p,s}$ become

$$R_{p,s} = + \left(\frac{2}{\pi} \right)^2 (-1)^p \left(\frac{\nu_s \sigma_s}{\eta_s} \right) \sin \pi \eta_s \sum_{n=1}^{\infty} \alpha^2 \frac{(-1)^n n^2}{\beta_n} \frac{\sin \pi n \alpha}{(n^2 \alpha^2 - p^2)(n^2 \alpha^2 - \eta_s^2 \alpha^2)} \tag{260}$$

Without the β_n , and using the summation formula, this becomes

$$\bar{R}_{p,s} = \left(\frac{2}{\pi} \right)^2 (-1)^p \left(\frac{\nu_s \sigma_s}{\eta_s} \right) \sin \pi \eta_s \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{\alpha^2 x^2 \sin \pi x(\alpha + 1) \cos 2\pi h x}{(\alpha^2 x^2 - p^2)(\alpha^2 x^2 - \eta_s^2 \alpha^2)} dx \tag{261}$$

Let $\alpha\eta_S = g_S$ and $y = \alpha x$.

$$\bar{R}_{p,s} = \left(\frac{2}{\pi}\right)^2 (-1)^p \left(\frac{\nu_S \sigma_S}{g_S}\right) \sin \pi \eta_S \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{y^2 \sin \pi \left(\frac{\alpha+1}{\alpha}\right) y \cos 2\pi \frac{h}{\alpha} y}{(y^2 - p^2)(y^2 - g_S^2)} dy \quad (262)$$

The integral is of the form

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon} \int_0^{\infty} \frac{y \sin \pi \epsilon y \sin \pi \left(\frac{\alpha+1}{\alpha}\right) y \cos 2\pi \frac{h}{\alpha} y}{(y^2 - p^2)(y^2 - g_S^2)} dy \quad (263)$$

Letting $u = \epsilon y$

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi} \int_0^{\infty} \frac{u \sin \pi u \sin \pi \left(\frac{\alpha+1}{\epsilon \alpha}\right) u \cos 2\pi \frac{h}{\alpha \epsilon} u}{\left(u^2 - \frac{p^2}{\epsilon^2}\right) \left(u^2 - \frac{g_S^2}{\epsilon^2}\right)} du \quad (264)$$

This procedure seems quite questionable. Therefore, go back to the previous expression and let

$$\left(\frac{\alpha+1}{\alpha}\right) y = v \quad (265)$$

$$\lim_{\epsilon \rightarrow 0} \frac{(\alpha+1)^2}{\alpha^2 \pi \epsilon} \int_0^{\infty} \frac{v \sin \pi \left(\frac{\epsilon \alpha}{1+\alpha}\right) v \sin \pi v \cos 2\pi \left(\frac{h}{\alpha+1}\right) v}{\left[v^2 - \frac{(\alpha+1)^2 p^2}{\alpha^2}\right] \left[v^2 - \frac{(\alpha+1)^2 g_S^2}{\alpha^2}\right]} dv \quad (266)$$

This gives $\bar{R}_{p,s}$ as

$$\bar{R}_{p,s} = \left(\frac{2}{\pi}\right)^2 (-1)^p \left(\frac{\nu_S \sigma_S}{\eta_S}\right) \sin \pi \eta_S \frac{(1+\alpha)}{\pi \alpha^2} \sum_{h=-\infty}^{\infty} \lim_{\epsilon \rightarrow 0} \frac{(\alpha+1)}{\alpha \epsilon} \int_0^{\infty} \frac{v \sin \left[\pi \left(\frac{\epsilon \alpha}{1+\alpha}\right) v \right] \sin \pi v \cos \left[2\pi \left(\frac{h}{\alpha+1}\right) v \right]}{\left[v^2 - \frac{(\alpha+1)^2 p^2}{\alpha^2}\right] \left[v^2 - \frac{(\alpha+1)^2 g_S^2}{\alpha^2}\right]} dv \quad (267)$$

The integral is in the form of Eq. 202 and gives for each h

$$\lim_{\beta \rightarrow 0} \frac{1}{\beta} I \left[\beta, \gamma = \frac{h}{1+\alpha} \quad m = \frac{(\alpha+1)p}{\alpha} \quad q = (\alpha+1)\eta_S \right] \quad (268)$$

Using the value of I from Eq. 214 shows that the terms vanish in pairs. A typical pair is

$$-Y(\beta - 1 - 2\gamma)\pi q + Y(\beta + 1 + 2\gamma)\pi q \quad (269)$$

since $Y(-x) = Y(x)$ (see Eq. 213). This expression vanishes when $\beta = 0$. Thus, the contribution of this pair of terms to $\frac{1}{\beta} I$ is

$$\frac{1}{4(q^2 - m^2)} \lim_{\beta \rightarrow 0} \frac{d}{d\beta} \left[-Y(\beta - 1 - 2\gamma)\pi q + Y(\beta + 1 + 2\gamma)\pi q \right] \quad (270)$$

Now evaluate

$$\begin{aligned}
& \frac{d}{d\beta} [-Y(\beta - 1 - 2\gamma)\pi q + Y(\beta + 1 + 2\gamma)\pi q] \\
&= \frac{d}{d\beta} [-\cos(\beta - 1 - 2\gamma)\pi q \operatorname{Ci}(\beta - 1 - 2\gamma)\pi q \\
&\quad - \sin(\beta - 1 - 2\gamma)\pi q \operatorname{Si}(\beta - 1 - 2\gamma)\pi q \\
&\quad + \cos(\beta + 1 + 2\gamma)\pi q \operatorname{Ci}(\beta + 1 + 2\gamma)\pi q \\
&\quad + \sin(\beta + 1 + 2\gamma)\pi q \operatorname{Si}(\beta + 1 + 2\gamma)\pi q] \\
&= [\sin(\beta - 1 - 2\gamma)\pi q \operatorname{Ci}(\beta - 1 - 2\gamma)\pi q \\
&\quad - \cos(\beta - 1 - 2\gamma)\pi q \operatorname{Si}(\beta - 1 - 2\gamma)\pi q \\
&\quad - \sin(\beta + 1 + 2\gamma)\pi q \operatorname{Ci}(\beta + 1 + 2\gamma)\pi q \\
&\quad + \cos(\beta + 1 + 2\gamma)\pi q \operatorname{Si}(\beta + 1 + 2\gamma)\pi q] \pi q \\
&\quad + \left[-\frac{\cos^2(\beta - 1 - 2\gamma)\pi q}{(\beta - 1 - 2\gamma)\pi q} - \frac{\sin^2(\beta - 1 - 2\gamma)\pi q}{(\beta - 1 - 2\gamma)\pi q} \right. \\
&\quad \left. + \frac{\cos^2(\beta + 1 + 2\gamma)\pi q}{(\beta + 1 + 2\gamma)\pi q} + \frac{\sin^2(\beta + 1 + 2\gamma)\pi q}{(\beta + 1 + 2\gamma)\pi q} \right] \pi q \\
&= [\sin(\beta - 1 - 2\gamma)\pi q \operatorname{Ci}(\beta - 1 - 2\gamma)\pi q - \cos(\beta - 1 - 2\gamma)\pi q \operatorname{Si}(\beta - 1 - 2\gamma)\pi q \\
&\quad - \sin(\beta + 1 + 2\gamma)\pi q \operatorname{Ci}(\beta + 1 + 2\gamma)\pi q + \cos(\beta + 1 + 2\gamma)\pi q \operatorname{Si}(\beta + 1 + 2\gamma)\pi q] \pi q \\
&\quad + \left[-\frac{1}{(\beta - 1 - 2\gamma)} + \frac{1}{(\beta + 1 + 2\gamma)} \right] \quad (271)
\end{aligned}$$

Thus the contribution of this pair of terms is

$$\frac{1}{4(q^2 - m^2)} \left[-2\pi q \sin(1 + 2\gamma) \operatorname{Ci}(1 + 2\gamma)\pi q + 2\pi q \cos(1 + 2\gamma)\pi q \operatorname{Si}(1 + 2\gamma)\pi q + \frac{2}{1 + 2\gamma} \right] \quad (272)$$

which may be written

$$\frac{1}{2(q^2 - m^2)(1 + 2\gamma)} [(1 + 2\gamma)\pi q \operatorname{Z}(1 + 2\gamma)\pi q + 1] \quad (273)$$

where

$$\operatorname{Z}(x) = \cos x \operatorname{Si} x - \sin x \operatorname{Ci} x \quad (274)$$

The whole integral can be written

$$\begin{aligned}
\int_{\beta=0}^1 \frac{1}{\beta} I(\beta, \gamma, m, q) &= \frac{1}{2(q^2 - m^2)} \left[\pi q \operatorname{Z}(1 + 2\gamma)\pi q + \frac{1}{1 + 2\gamma} - \pi m \operatorname{Z}(1 + 2\gamma) - \frac{1}{1 + 2\gamma} \right. \\
&\quad \left. + \pi q \operatorname{Z}(1 - 2\gamma)\pi q + \frac{1}{1 - 2\gamma} - \pi m \operatorname{Z}(1 - 2\gamma) - \frac{1}{1 - 2\gamma} \right] \quad (275)
\end{aligned}$$

which reduces to

$$\frac{\pi}{2(q^2 - m^2)} \{q[\operatorname{Z}(1 + 2\gamma)\pi q + \operatorname{Z}(1 - 2\gamma)\pi q] - m[\operatorname{Z}(1 + 2\gamma) + \operatorname{Z}(1 - 2\gamma)]\} \quad (276)$$

These results give for $\bar{R}_{p,s}$

$$\begin{aligned}
\bar{R}_{p,s} &= \left(\frac{2}{\pi}\right)^2 (-1)^p \left(\frac{\nu_s \sigma_s}{\eta_s}\right) \sin \pi \eta_s \frac{1}{\pi \alpha^2} \sum_{h=-\infty}^{\infty} \frac{\pi}{2} \frac{1}{(\eta_s^2 - \frac{p^2}{\alpha^2})} \left[\eta_s \left[\operatorname{Z}\left[\left(1 + \frac{2h}{1 + \alpha}\right) \pi(\alpha + 1)\eta_s\right] \right. \right. \\
&\quad \left. \left. + \operatorname{Z}\left[\left(1 - \frac{2h}{1 + \alpha}\right) \pi(\alpha + 1)\eta_s\right] \right] - \frac{p}{\alpha} \left[\operatorname{Z}\left[\left(1 + \frac{2h}{1 + \alpha}\right) \pi(\alpha + 1)\frac{p}{\alpha}\right] + \operatorname{Z}\left[\left(1 - \frac{2h}{1 + \alpha}\right) \pi(\alpha + 1)\frac{p}{\alpha}\right] \right] \right] \quad (277)
\end{aligned}$$

which may be written

$$\bar{R}_{p,s} = \frac{2}{\pi^2} (-1)^p \left(\frac{\nu_s \sigma_s}{g_s} \right) \sin \pi \eta_s \frac{1}{(g_s^2 - p^2)} \sum_{h=-\infty}^{\infty} \left[g_s \{ Z[\pi \eta_s (1 + \alpha + 2h)] \right. \\ \left. + Z[\pi \eta_s (1 + \alpha - 2h)] \} - p \left\{ Z\left[\pi \frac{p}{\alpha} (1 + \alpha + 2h)\right] + Z\left[\pi \frac{p}{\alpha} (1 + \alpha - 2h)\right] \right\} \right] \quad (278)$$

This may also be written

$$\bar{R}_{p,s} = \left(\frac{2}{\pi} \right)^2 (-1)^p \left(\frac{\nu_s \sigma_s}{g_s} \right) \sin \pi \eta_s \frac{1}{(p^2 - g_s^2)} \left[\left\{ p Z\left[\pi \frac{p}{\alpha} (1 + \alpha)\right] - g_s Z[\pi \eta_s (1 + \alpha)] \right\} \right. \\ \left. + \sum_{h=1}^{\infty} \left\{ p \left\{ Z\left[\pi \frac{p}{\alpha} (1 + \alpha + 2h)\right] + Z\left[\pi \frac{p}{\alpha} (1 + \alpha - 2h)\right] \right\} \right. \right. \\ \left. \left. - g_s \{ Z[\pi \eta_s (1 + \alpha + 2h)] + Z[\pi \eta_s (1 + \alpha - 2h)] \} \right\} \right] \quad (279)$$

Now evaluate the $\bar{M}_p''$ in a similar way.

$$\bar{M}_p'' = - \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{\pi \sin \pi \left(\frac{\alpha + 1}{\alpha} \right) y \cos 2\pi \frac{h}{\alpha} y}{(y^2 - p^2)} dy \quad (280)$$

Let $x = \left(\frac{\alpha + 1}{\alpha} \right) y$.

$$\bar{M}_p'' = - \left(\frac{2}{\pi} \right)^2 (-1)^p \sum_{h=-\infty}^{\infty} \int_0^{\infty} \frac{(\alpha + 1)\pi \sin \pi x \cos \frac{2\pi h}{(1 + \alpha)} x}{\alpha \left[x^2 - \frac{(\alpha + 1)^2 p^2}{\alpha^2} \right]} dx \quad (281)$$

This may be written in the form of Eq. 202.

$$\bar{M}_p'' = - \left(\frac{2}{\pi} \right)^2 (-1)^p \left(\frac{\alpha + 1}{\alpha} \right) \sum_{h=-\infty}^{\infty} \lim_{\substack{\beta \rightarrow 0 \\ q \rightarrow 0}} \frac{1}{\beta} \int_0^{\infty} \frac{x \sin \pi \beta x \sin \pi x \cos \frac{2\pi h}{(1 + \alpha)} x}{\left[x^2 - \frac{(\alpha + 1)^2 p^2}{\alpha^2} \right] (x^2 - q^2)} dx \quad (282)$$

Using the previous results (see the determination of $\bar{R}$) gives

$$\bar{M}_p'' = \left(\frac{2}{\pi} \right) (-1)^p \left(\frac{\alpha + 1}{\alpha} \right) \sum_{h=-\infty}^{\infty} \lim_{q \rightarrow 0} \left[\frac{1}{(m^2 - q^2)} [qZ(1 + 2\gamma)\pi q + qZ(1 - 2\gamma)\pi q \right. \\ \left. - mZ(1 + 2\gamma)\pi m - mZ(1 - 2\gamma)\pi m \right] \quad (283)$$

$$\text{where } m = \frac{(\alpha + 1)p}{\alpha} \\ \gamma = \frac{h}{\alpha + 1}$$

In order to find the $\lim q \rightarrow 0$, consider

$$qZ\pi kq = q (\cos \pi kq - \sin \pi kq \operatorname{Ci} \pi kq) \lim_{q \rightarrow 0} qZ(\pi kq) = \pi kq^2 - q^2 (C + \ln \pi kq) = -q^2 \ln \pi kq \rightarrow 0 \quad (284)$$

Thus,

$$\bar{M}_p'' = -(-1)^p \left(\frac{\alpha + 1}{\alpha} \right) \left(\frac{\pi}{2} \right) \frac{\alpha}{(\alpha + 1)p} \left(\frac{2}{\pi} \right)^2 \sum_{h=-\infty}^{\infty} \left[Z\pi \left(1 + \frac{2h}{\alpha + 1} \right) \frac{(\alpha + 1)p}{\alpha} + Z\pi \left(1 - \frac{2h}{\alpha + 1} \right) \frac{(\alpha + 1)p}{\alpha} \right]$$

or

$$\bar{M}_p'' = \left(\frac{2}{\pi} \right) \left(\frac{-1}{p} \right) (-1)^p \sum_{h=-\infty}^{\infty} \left[Z(\alpha + 1 + 2h) \frac{\pi p}{\alpha} + Z(\alpha + 1 - 2h) \frac{\pi p}{\alpha} \right] \quad (285)$$

Conclusion

This concludes the analysis for the field due to cylindrical poles magnetized by thin coils so that a known potential distribution is present over the pole sides and the yoke surfaces.

Enough numerical work has been done with this material (by George Hinman) to show that it gives reasonable results. The labor involved indicates that the application of this work, if any, should be in fields other than magnet design. In principle, this work could be extended by the methods described in the subsections, Square Equipotential and Poles Obtained by Superposition, of Sec. 1.1.

APPENDIX III

POLE TIP WITH COIL AND YOKE

It has been assumed in Sec. 1.1 that the influence of the coil and the yoke could be approximately neglected, in studying pole tips. This is not too serious in magnets of the type now constructed, but in much larger machines it would be desirable to bring the coil much closer to the gap. The purpose of the following remarks is merely to indicate that a two-dimensional analysis of such pole tips is quite feasible.

The imaginary part of the function

$$V = \frac{1}{\pi} \log \left[\left(1 + \frac{1}{3} e^{\pi z/5} \right)^4 \left(1 + 2e^{\pi z/5} \right) \right] \quad (286)$$

gives the field developed by a pole which is four half-gaps high and which is excited by a single wire lying on the yoke. The constants are so chosen that the pole comes straight out of the yoke as far as is possible and then becomes asymptotic to the nominal half-gap. Figure 1.40 is an accurate sketch of the $V = 1$ and $V = 0$ equipotentials for the imaginary part of Eq. 1.

The field at the mid-plane is

$$H = \frac{2e^{\pi x/5} (1 + e^{\pi x/5})}{(3 + e^{\pi x/5}) (1 + 2e^{\pi x/5})} \quad (287)$$

This field shape is far from satisfactory and would require alteration of the type discussed in the subsection, Poles Obtained by Superposition, Sec. 1.1.

APPENDIX IV

SEARCH COILS TO MEASURE FIELD AT A POINT

Design of Search Coils, Introduction

Pick-up coils or search coils are used for various purposes, to measure magnetic potential difference, to measure total flux linking some contour, to measure the field at a point, etc.

The properties which the coil should have depend on what is to be measured and the detecting instrument to which the coil is connected. The important properties are the geometry, the size, the turns-area, and the resistance. For example, for the measurement of potential difference, a long coil (wound on a threaded form if high accuracy is required), as small in diameter as possible, is used. Apart from physical limitations of size, the limiting diameter is determined by the turns-area and conductance which the coil must have.

The coils considered below are designed to measure the field at a point. The difficulty encountered is similar to that encountered with a potential measuring coil. The ideal coil is so small that no appreciable variation in the field occurs across it. However, the turns-area and conductance required may make it necessary to build larger coils. Under these circumstances careful consideration of the geometry of the coil is required. Also, where variation in coil resistance with temperature introduces errors, it may be desirable to use manganin or some similar conductor in the coils.

Two types of coils are considered, the first, which is capable of greater precision, is wound in conical grooves cut in a sphere. It is difficult to construct, but two such coils 0.64 in. in diameter have been built.

The second type of coil is simply a spool, with suitable dimensions, which is wound with wire. It is clearly much more easily constructed.

A table of the first associated Legendre polynomials, which is useful in designing search coils, is given at the end of this appendix.

An Accurate Search Coil to Measure the Field at a Point

When no sources are present in a sphere of radius c , the radial component of the field within the sphere may be written with sufficient generality for the present purpose.

$$H_\rho = \sum_{m=1}^{\infty} A_m \left(\frac{\rho}{c}\right)^{m-1} P_m(\cos \theta) \quad (288)$$

where A_m = set of constants depending on the field

ρ = radial variable (spherical coordinates)

P_m = Legendre functions

θ = polar-angle variable

Clearly the field along the polar axis at the origin is A_1 . If the pick-up coil detects other coefficients, it does not accurately measure the field at the center.

The flux passing through a loop of wire lying in a sphere of radius $b < c$ at an angle α is

$$\phi = 2\pi b^2 \sum_{m=1}^{\infty} A_m \left(\frac{b}{c}\right)^{m-1} \int_0^\alpha \sin \theta P_m(\cos \theta) d\theta \quad (289)$$

The indicated integration gives

$$\phi = 2\pi b^2 \sum_{m=1}^{\infty} A_m \left(\frac{b}{c}\right)^{m-1} \frac{\sin \alpha}{m(m+1)} P_m^1(\cos \alpha) \quad (290)$$

In order to eliminate the coefficients of $A_2, A_4, \dots$ the pick-up coil may be made symmetrical about the equatorial plane. The flux linkage for such an arrangement may be written

$$W_1 = 4\pi n b^2 \sum_{m=0}^{\infty} \frac{A(2m+1) \left(\frac{b}{c}\right)^{2m}}{(2m+1)(2m+2)} \sin \alpha P_{(2m+1)}^1 (\cos \alpha) \quad (291)$$

where n = number of turns in each of the two coils. With p such pairs of coils the linkage is

$$W = 4\pi \sum_{i=1}^p n_i b_i^2 \sum_{m=0}^{\infty} \frac{A(2m+1) \left(\frac{b_i}{c}\right)^{2m}}{(2m+1)(2m+2)} \sin \alpha_i P_{(2m+1)}^1 (\cos \alpha_i) \quad (292)$$

The first term of this series gives the turns-area of the arrangement (times the field, A_1),

$$W^1 = 2\pi \sum_{i=1}^p A_1 n_i b_i^2 \sin^2 \alpha_i \quad (293)$$

The particular case where $p = 2$ will now be considered in some detail. There are four variables of interest,

$$\frac{b_2}{b_1} = B \quad \frac{n_2}{n_1} = N \quad \alpha_1 \quad \alpha_2 \quad (294)$$

By appropriate adjustment of these parameters it should be possible to cause the coefficients of A_3, A_5, A_7 , and A_9 in Eq. 292 to vanish. These coefficients are (with a factor $\frac{4\pi}{c^{2m}(2m+1)(2m+2)}$ suppressed)

$$\begin{aligned} & \sum_{i=1}^2 n_i b_i^4 \sin \alpha_i P_3^1 (\cos \alpha_i) \\ & \sum_{i=1}^2 n_i b_i^6 \sin \alpha_i P_5^1 (\cos \alpha_i) \\ & \sum_{i=1}^2 n_i b_i^8 \sin \alpha_i P_7^1 (\cos \alpha_i) \\ & \sum_{i=1}^2 n_i b_i^{10} \sin \alpha_i P_9^1 (\cos \alpha_i) \end{aligned} \quad (295)$$

respectively. The analytic approach to this problem involves equations of a high order; thus a numerical method was used. It is unfortunate that no set of the parameters of Eq. 294 exists, which causes all the members of 2.8 to vanish simultaneously. Since there are four parameters, the first three members of Eq. 295 may be made to vanish in many ways.

A very desirable solution is to have $n_2/n_1 = N$ a simple rational number. Such a solution certainly exists but has not been found. There are two designs, which are easy to determine, and these are given below.

$P_7^1 (\cos \theta) = 0$ has roots at $\theta = 53.72$ and 77.92 deg. The signs of P_3^1, P_5^1 , and P_9^1 are different at these angles, thus the coefficients of A_3 and A_5 (see Eqs. 295) can be eliminated, and the coefficient A_9 will be "small." This solution is

$$\begin{aligned} \alpha_1 &= 53.72^\circ & \frac{b_{78^\circ}}{b_{54^\circ}} &= 1.800 \\ \alpha_2 &= 77.92^\circ & \frac{n_{54^\circ}}{n_{78^\circ}} &= 16.33 \end{aligned} \quad (296)$$

This is not a good coil for most purposes, since the coil with the greater number of turns has the smaller radius.

The other arrangement which is simple to calculate gives a good design. It has coils placed at the roots of P_5^1 . At the angles where P_5^1 vanishes, $\arccos x_1$ and $\arccos x_2$, then

$$\frac{d}{dx} P_5(x_n) = 0 \quad (297)$$

In the recursion relation,

$$\frac{d}{dx} P_{m+1}(x) - \frac{d}{dx} P_{m-1}(x) = (2m+1) P_m(x) \quad (298)$$

Equation 297 gives

$$\begin{aligned} 9P_4(x_n) &= \frac{d}{dx} P_5(x_n) - \frac{d}{dx} P_3(x_n) = -\frac{d}{dx} P_3(x_n) \\ 13P_6(x_n) &= \frac{d}{dx} P_7(x_n) - \frac{d}{dx} P_5(x_n) = \frac{d}{dx} P_7(x_n) \end{aligned} \quad (299)$$

Another recursion relation is

$$(1-x^2) \frac{d}{dx} P_m(x) = \frac{m(m+1)}{(2m+1)} [P_{m-1}(x) - P_{m+1}(x)] \quad (300)$$

in which Eq. 297 gives

$$P_4(x_n) = P_6(x_n) \quad (301)$$

Equations 299 and 301 give

$$\frac{d}{dx} P_3(x_n) = -\frac{9}{13} \frac{d}{dx} P_7(x_n)$$

or

$$(1-x_n^2) \frac{d}{dx} P_3(x_n) = -\frac{9}{13} (1-x_n^2) \frac{d}{dx} P_7(x_n)$$

which is identical with

$$(1-x_n^2)^{1/2} P_3^1(x_n) = -\frac{9}{13} (1-x_n^2)^{1/2} P_7^1(x_n) \quad (302)$$

This is true clearly for both values of x_n , x_1 and x_2 , where $P_5^1(x)$ vanishes. From Eq. 302 and the condition that the first three expressions of Eq. 295 should vanish, the ratio of the radii of the coils is found to be

$$B^4 = \frac{(1-x_2^2)^{1/2} P_3^1(x_2)}{(1-x_1^2)^{1/2} P_3^1(x_1)} \cdot \frac{(1-x_1^2)^{1/2} P_7^1(x_1)}{(1-x_2^2)^{1/2} P_7^1(x_2)} = 1 \quad (303)$$

The ratio of the number of turns from the same information is

$$N = \frac{n_2}{n_1} = - \frac{(1 - x_1^2)^{1/2} P_3^1(x_1)}{(1 - x_1^2)^{1/2} P_3^1(x_2)} = - \frac{(1 - x_1^2)^{1/2} P_7^1(x_1)}{(1 - x_2^2)^{1/2} P_7^1(x_2)} \quad (304)$$

The numerical data are

$$\left. \begin{aligned} B = \frac{b_2}{b_1} = 1 \quad N = \frac{n_2}{n_1} = 1.4660 = - \frac{1 + 2\sqrt{7}}{1 - 2\sqrt{7}} \\ \theta_1 = 40.09 \text{ deg} \quad \theta_2 = 73.43 \text{ deg} \quad \cos^2 \theta = \frac{1}{3} \left(1 + \frac{2}{\sqrt{7}} \right) \end{aligned} \right\} \quad (305)$$

Such a coil has been constructed, but it is very difficult to check its accuracy experimentally. The errors have therefore been estimated by integrating the flux linkages over a coil of width $2\delta_1$ centered at 40.09 deg and another of width $n_2/n_1 \cdot 2\delta_1$ centered at 73.43 deg . This coil is illustrated in Fig. 1.41. Only the leading term was kept, and each was divided by the coefficient of A_1 and by A_1 . The results were

$$\left. \begin{aligned} E_1 &= 1 \\ E_3 &= 0.2 \quad \frac{A_3}{A_1} \left(\frac{b_1}{c} \right)^2 \delta_1^2 \\ E_5 &= -0.002 \quad \frac{A_5}{A_1} \left(\frac{b_1}{c} \right)^4 \delta_1^2 \\ E_7 &= -0.4 \quad \frac{A_7}{A_1} \left(\frac{b_1}{c} \right)^6 \delta_1^2 \\ E_9 &= -0.05 \quad \frac{A_9}{A_1} \left(\frac{b_1}{c} \right)^8 \end{aligned} \right\} \quad (306)$$

Pick-up Coil Convenient to Wind That Measures the Field at a Point

A convenient form of pick-up coil for measuring the field at a point is one where one size of wire is wound on a cylindrical spool of the general form shown in Fig. 1.42.

There are three parameters, x_2/x_1 , y_1/x_1 , and y_2/x_1 , which can be used to adjust the characteristics of the coil. For this reason, one might expect that the coefficients of A_3 , A_5 , and A_7 could be made to vanish simultaneously. However, it is shown that the coefficients of A_3 and A_5 cannot be simultaneously zero.

The flux associated with A_m which is linked by an element of area in the coil is

$$m(m+1) \frac{c^{m-1}}{4\pi} \frac{d\phi_m}{A_m} = \frac{b(m+1) \sin \alpha P_m^1(\cos \alpha) da}{b(m+1) \sin^2 \alpha P_m'(\cos \alpha) da} \quad (307)$$

where $' = \frac{d}{d(\cos \theta)}$, and b, α denote position of area da , spherical coordinates.

The right-hand side may be written in terms of Cartesian coordinates x, y for $m = 1, 3, 5$.

$$\begin{aligned} m = 1 &= x^2 da \\ m = 3 &= (4x^2y^2 - x^4) da \\ m = 5 &= (8x^2y^4 - 12y^2x^4 + x^6) da \end{aligned} \quad (308)$$

and these functions integrated over the area of the coil give

$$\begin{aligned}
m = 1 &= \frac{1}{3} (x_2^3 - x_1^3)(y_2 - y_1) \\
m = 3 &= \frac{4}{9} (x_2^3 - x_1^3)(y_2^3 - y_1^3) - \frac{1}{5} (x_2^5 - x_1^5)(y_2 - y_1) \\
m = 5 &= \frac{8}{15} (x_2^3 - x_1^3)(y_2^5 - y_1^5) - \frac{4}{5} (x_2^5 - x_1^5)(y_2^3 - y_1^3) + \frac{1}{7} (x_2^7 - x_1^7)(y_2 - y_1)
\end{aligned} \tag{309}$$

By setting the coefficients of A_3 and A_5 equal to zero and removing the common factor $(x_2 - x_1)(y_2 - y_1)$ gives:

the coefficient of A_3 vanishes if

$$0 = \frac{4}{9} (x_2^2 + x_1x_2 + x_1^2)(y_2^2 + y_1y_2 + y_1^2) - \frac{1}{5} (x_2^4 + x_2^3x_1 + x_2^2x_1^2 + x_2x_1^3 + x_1^4) \tag{310}$$

the coefficient of A_5 vanishes if

$$\begin{aligned}
0 = \frac{8}{15} (x_2^2 + x_2x_1 + x_1^2)(y_2^4 + y_2^3y_1 + y_2^2y_1^2 + y_2y_1^3 + y_1^4) - \frac{4}{5} (x_2^4 + x_2^3x_1 + x_2^2x_1^2 + x_2x_1^3 + x_1^4) \\
(y_2^2 + y_2y_1 + y_1^2) + \frac{1}{7} (x_2^6 + x_2^5x_1 + x_2^4x_1^2 + x_2^3x_1^3 + x_2^2x_1^4 + x_2x_1^5 + x_1^6)
\end{aligned}$$

To simplify the appearance of these equations, let $x_2 = x \geq 1$ and $x_1 = 1$ merely to set the scale, and let

$$\begin{aligned}
E_2 &= x^2 + x + 1 \\
E_4 &= x^4 + x^3 + x^2 + x + 1 \\
E_6 &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
C_2 &= C^2 + C + 1 \\
C_4 &= C^4 + C^3 + C^2 + C + 1 \\
C &= \frac{y_2}{y_1} \neq \text{c of Eqs. 307} \\
d &= y_1^2
\end{aligned}$$

Equation 310 now reads

for A_3

$$0 = \frac{4}{9} E_2 C_2 d - \frac{1}{5} E_4$$

for A_5

$$0 = \frac{8}{15} E_2 C_4 d^2 - \frac{4}{5} E_4 C_2 d + \frac{1}{7} E_6$$

To show that these cannot be satisfied simultaneously, solve the A_3 equation for d and eliminate d in the A_5 equation.

$$\begin{aligned}
d &= \frac{9}{20} \frac{E_4}{E_2 C_2} \\
0 &\stackrel{?}{=} \frac{8}{15} \frac{9^2}{20^2} \frac{E_4^2 C_4}{E_2^2 C_2^2} - \frac{4 \cdot 9}{5 \cdot 20} \frac{E_4^2}{E_2} + \frac{1}{7} E_6
\end{aligned} \tag{312}$$

The question mark is placed over the equals sign since it may not be possible to make the right-hand side of this equation vanish.

The E's and C's have a lower bound of three, so that Eq. 312 may be divided by $(4 \cdot 9/5 \cdot 20)$ (E_4^2/E_2) and rearranged to read

$$1 \stackrel{?}{=} \frac{5 \cdot 20 \cdot 8 \cdot 9^2}{15 \cdot 4 \cdot 9 \cdot 20^2} \frac{C_4}{C_2^2} + \frac{5 \cdot 20}{4 \cdot 9 \cdot 7} \frac{E_6 E_2}{E_4^2}$$

or

$$1 \stackrel{?}{=} \frac{1}{630} \left(21 \frac{C_4}{C_2^2} + 250 \frac{E_6 E_2}{E_4^2} \right) \quad (313)$$

By expanding C_4 , C_2^2 , E_6 , E_2 , and E_4^2 , it can be shown that

$$\frac{C_4}{C_2^2} \leq 1$$

and

$$\frac{E_6 E_2}{E_4^2} \leq 1 \quad (314)$$

Thus Eq. 313 may be written

$$1 \stackrel{?}{\leq} \frac{271}{630} \quad (315)$$

Because the inequality of Eq. 315 is not true, there is no arrangement in this general class which will simultaneously eliminate coefficients of A_3 and A_5 .

Since the coefficient of A_5 cannot be eliminated, the best remaining solution is that one which detects the minimum A_5 . To find this arrangement, the ratio of ϕ_5 to ϕ_1 was determined from Eqs. 307 and 309 and minimized. ϕ_1 and ϕ_5 are given by

$$\begin{aligned} \phi_1 &= \frac{4\pi A_1}{2 \cdot 3} (x_2^3 - x_1^3)(y_2 - y_1) \\ \phi_5 &= \frac{4\pi A_5}{c^4 6 \cdot 5} \left[\frac{8}{15} (x_2^3 - x_1^3)(y_2^5 - y_1^5) - \frac{4}{5} (x_2^5 - x_1^5)(y_2^3 - y_1^3) + \frac{1}{7} (x_1^7 - x_2^7)(y_2 - y_1) \right] \end{aligned} \quad (316)$$

The ratio in the notation of Eq. 311 is

$$\frac{\phi_5}{\phi_1} = \frac{A_5}{5c^4 A_1} \left(\frac{\frac{8}{15} E_2 C_4 d^2 - \frac{4}{5} E_4 C_2 d + \frac{1}{7} E_6}{E_2} \right) \quad (317)$$

Using the condition that the coefficient of A_3 vanishes to eliminate d gives

$$\frac{5c^4 A_1 \phi_5}{A_5 \phi_1} = \frac{3^2 E_4^2}{5^2 E_2^2} \left(\frac{3}{2 \cdot 5} \frac{C_4}{C_2^2} - 1 \right) + \frac{1}{7} \frac{E_6}{E_2} \quad (318)$$

Setting the derivative, with respect to $C = 0$, gives

$$\left(\frac{E_4^2}{E_2^2} \frac{C^6 + 4C^5 + 4C^4 - 4C^2 - 4C + 1}{C_2^4} \right) = 0 \quad (319)$$

The roots are $C = 1$ and $C = \infty$.

With these roots (i.e., $C = 1$, $C = \infty$) the equation obtained by differentiating with respect to x has no roots. It turns out that the extreme design $\left(y_1 = y_2 = \frac{x_2}{2} \sqrt{\frac{3}{5}}, x_1 = 0 \right)$ has the smallest ratio of the coefficient of A_5 to the coefficient of A_1 .

The equation which the coil should satisfy is the first of Eqs. 310.

$$\frac{9}{20} \frac{(x_2^4 + x_2^3 x_1 + x_2^2 x_1^2 + x_2 x_1^3 + x_1^4)}{(x_2^2 + x_2 x_1 + x_1^2)} = y_2^2 + y_2 y_1 + y_1^2 \quad (320)$$

so that the coefficient of A_3 is eliminated.

The following quantities are of interest:

1. Resistance per turn squared

$$\frac{R}{N^2} = \frac{\pi}{2} \frac{(x_2 + x_1) \rho}{(y_2 - y_1)(x_2 - x_1)} \quad (321)$$

2. Area per turn

$$\frac{A}{N} = \frac{\pi}{3} (x_2^2 + x_2 x_1 + x_1^2) \quad (322)$$

3. $\frac{\text{Resistance}}{(\text{turns-area})^2}$

$$\frac{R}{A^2} = \frac{9}{2\pi} \frac{\rho}{(y_2 - y_1)} \frac{(x_2^2 - x_1^2)}{(x_2^3 - x_1^3)^2} \quad (323)$$

4. Table of

$$\frac{2m+1}{m(m+1)} \sin \theta P_m^1(\cos \theta)$$

The following table gives the values of

$$B_m^1 = \frac{2m+1}{m(m+1)} \sin \theta P_m^1(\cos \theta)$$

$m = 3, 5, 7, 9$

useful in designing or checking the previously described search coils and others.

The tables were made from existing* tables of $P_m(\cos \theta)$ by the recursion formula

$$B_m^1 = P_{m-1} - P_{m+1}$$

The tables were carefully made and checked manually.

*W. Lane and D. Sweeney, Table of Legendre Polynomials $P_n(\cos \theta)$ for $n = 0$ (1) 20, and $\theta = 0^\circ$ (1) 180° to Six Decimals, LADC-361, (1947).

Table 1.3

θ	B_3^1	B_5^1	B_7^1	B_9^1
0	0.000000	0.000000	0.000000	0.000000
1	0.001064	0.001672	0.002279	0.002883
2	0.004256	0.006671	0.009062	0.011418
3	0.009554	0.014920	0.020164	0.025244
4	0.016927	0.026307	0.035305	0.043789
5	0.026334	0.040674	0.054090	0.066280
6	0.037720	0.057815	0.076024	0.091760
7	0.051017	0.077487	0.100524	0.119135
8	0.066151	0.099376	0.126942	0.147210
9	0.083031	0.123278	0.154563	0.174728
10	0.101559	0.148747	0.182625	0.200408
11	0.121628	0.175424	0.210345	0.223012
12	0.143120	0.202946	0.236930	0.241369
13	0.165906	0.230893	0.261586	0.254423
14	0.189855	0.258851	0.283556	0.261281
15	0.214823	0.286388	0.302122	0.261242
16	0.240661	0.313081	0.316624	0.253819
17	0.267215	0.338503	0.326479	0.238779
18	0.294325	0.362241	0.331194	0.216137
19	0.321828	0.383891	0.330375	0.186174
20	0.349556	0.403075	0.323742	0.149430
21	0.377337	0.419430	0.311137	0.106697
22	0.405000	0.432632	0.292530	0.058993
23	0.432373	0.442379	0.268006	0.007532
24	0.459284	0.448414	0.237797	-0.046298
25	0.485558	0.450515	0.202247	-0.100991
26	0.511029	0.448505	0.161831	-0.154957
27	0.535525	0.442254	0.117114	-0.206572
28	0.558885	0.431677	0.068860	-0.254240
29	0.580949	0.416741	0.017777	-0.296433
30	0.601562	0.397462	-0.035247	-0.331737
31	0.620578	0.373906	-0.089292	-0.358917
32	0.637853	0.346193	-0.143385	-0.376937
33	0.653254	0.314490	-0.196533	-0.385015
34	0.666655	0.279012	-0.247754	-0.382620
35	0.677939	0.240023	-0.296008	-0.369597
36	0.687001	0.197833	-0.340394	-0.345996
37	0.693744	0.152789	-0.379993	-0.312244
38	0.698078	0.105276	-0.413969	-0.269058
39	0.699932	0.055719	-0.441584	-0.217457
40	0.699240	0.004567	-0.462198	-0.158718
41	0.695954	-0.047700	-0.475291	-0.094362
42	0.690031	-0.100590	-0.480472	-0.026090
43	0.681447	-0.153587	-0.477491	+0.044237
44	0.670178	-0.206170	-0.466243	0.114671

Table 1.3—(Continued)

θ	B_3^1	B_5^1	B_7^1	B_9^1
45	0.656250	-0.257812	-0.446778	0.183228
46	0.639649	-0.307989	-0.419293	0.247933
47	0.620409	-0.356182	-0.384137	0.306900
48	0.598569	-0.401889	-0.341811	0.358375
49	0.574177	-0.444619	-0.292945	0.400783
50	0.547299	-0.483913	-0.238304	0.432796
51	0.518009	-0.519335	-0.178769	0.453357
52	0.486397	-0.550486	-0.115319	0.461721
53	0.452560	-0.577003	-0.049020	0.457484
54	0.416611	-0.598570	+0.018998	0.440600
55	0.378671	-0.614911	0.087558	0.411370
56	0.338874	-0.625803	0.155458	0.370462
57	0.297358	-0.631076	0.221502	0.318874
58	0.254277	-0.630615	0.284503	0.257930
59	0.209789	-0.624357	0.343321	0.189222
60	0.164063	-0.612305	0.396880	0.114591
61	0.117271	-0.594512	0.444190	0.036052
62	0.069591	-0.571089	0.484357	-0.044244
63	0.021213	-0.542210	0.516609	-0.124070
64	-0.027676	-0.508098	0.540310	-0.201204
65	-0.076881	-0.469037	0.554971	-0.273467
66	-0.126206	-0.425356	0.560261	-0.338812
67	-0.175452	-0.377437	0.556012	-0.395362
68	-0.224422	-0.325700	0.542225	-0.441480
69	-0.272916	-0.270612	0.519071	-0.475806
70	-0.320734	-0.212677	0.486893	-0.497308
71	-0.367681	-0.152423	0.446188	-0.505299
72	-0.413564	-0.090411	0.397616	-0.499476
73	-0.458192	-0.027217	0.341974	-0.479920
74	-0.501380	+0.036563	0.280194	-0.447095
75	-0.542948	0.100328	0.213322	-0.401852
76	-0.582724	0.163477	0.142490	-0.345374
77	-0.620538	0.225402	0.068921	-0.279188
78	-0.656232	0.285516	-0.006125	-0.205078
79	-0.689657	0.343241	-0.081343	-0.125076
80	-0.720672	0.398024	-0.155430	-0.041374
81	-0.749144	0.449334	-0.227093	+0.043719
82	-0.774955	0.496680	-0.295082	0.127851
83	-0.797992	0.439599	-0.358204	0.208687
84	-0.818160	0.577677	-0.415352	0.283981
85	-0.835373	0.610547	-0.465526	0.351641
86	-0.849557	0.637888	-0.507842	0.409780
87	-0.860654	0.659435	-0.541553	0.456774
88	-0.868612	0.674975	-0.566064	0.491310
89	-0.873401	0.684361	-0.580947	0.512426
90	-0.875000	0.687500	-0.585938	0.519532

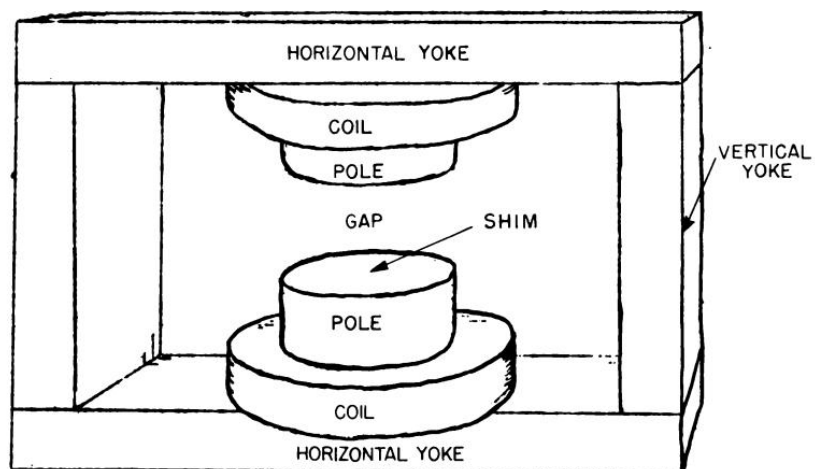


Fig. 1.1 — Parts of cyclotron magnet, gap greatly exaggerated.

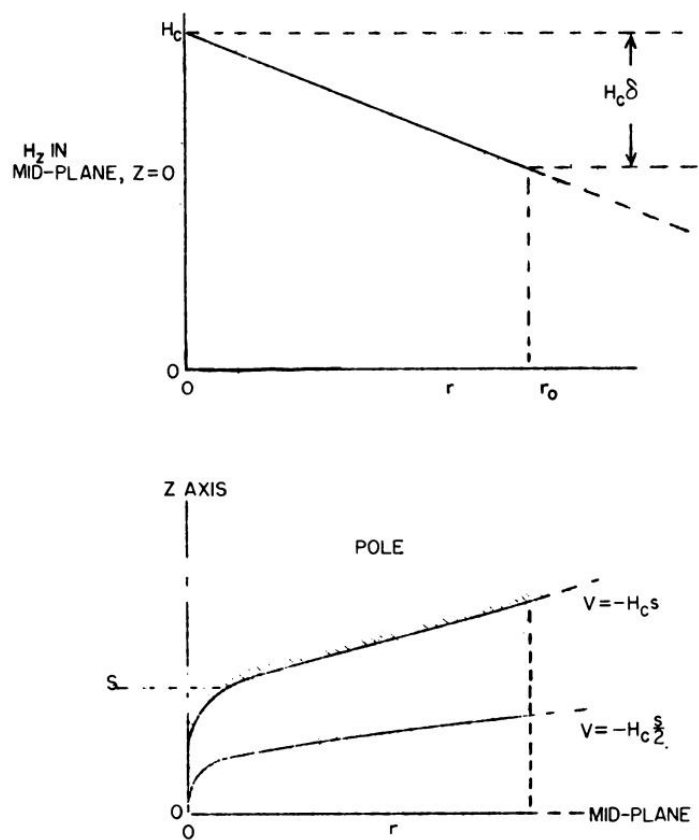


Fig. 1.2 — Field shape and coordinates used in shim calculation.

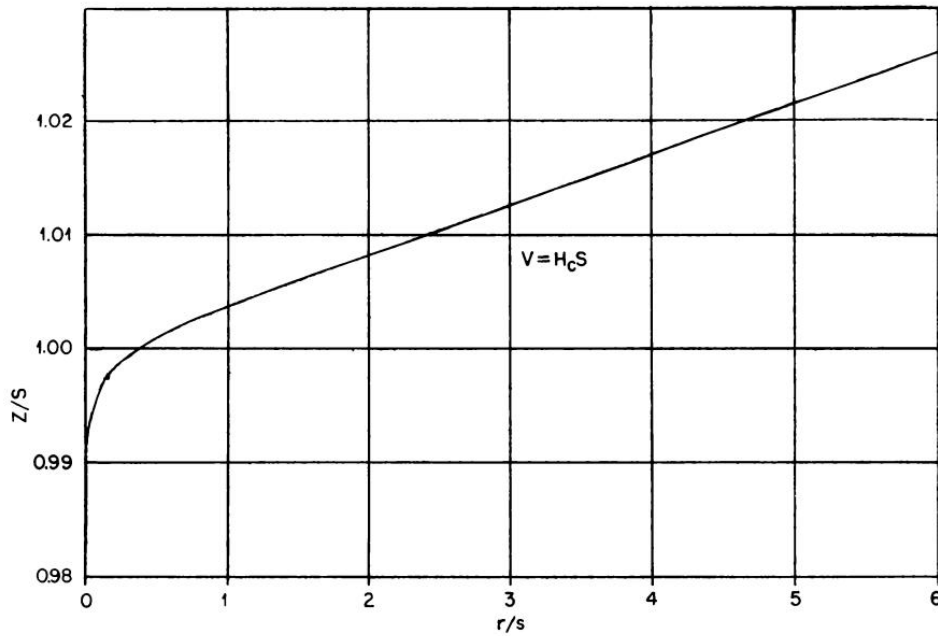


Fig. 1.3—Shape of an equipotential which decreases the field linearly to 96 per cent at the radius $r = 9.28s$. A spike of extremely small radius (not shown) crosses the gap at $r = 0$. r , radius; z , distance from mid-plane; s , nominal half-gap; H_C , field at $r = 0$.

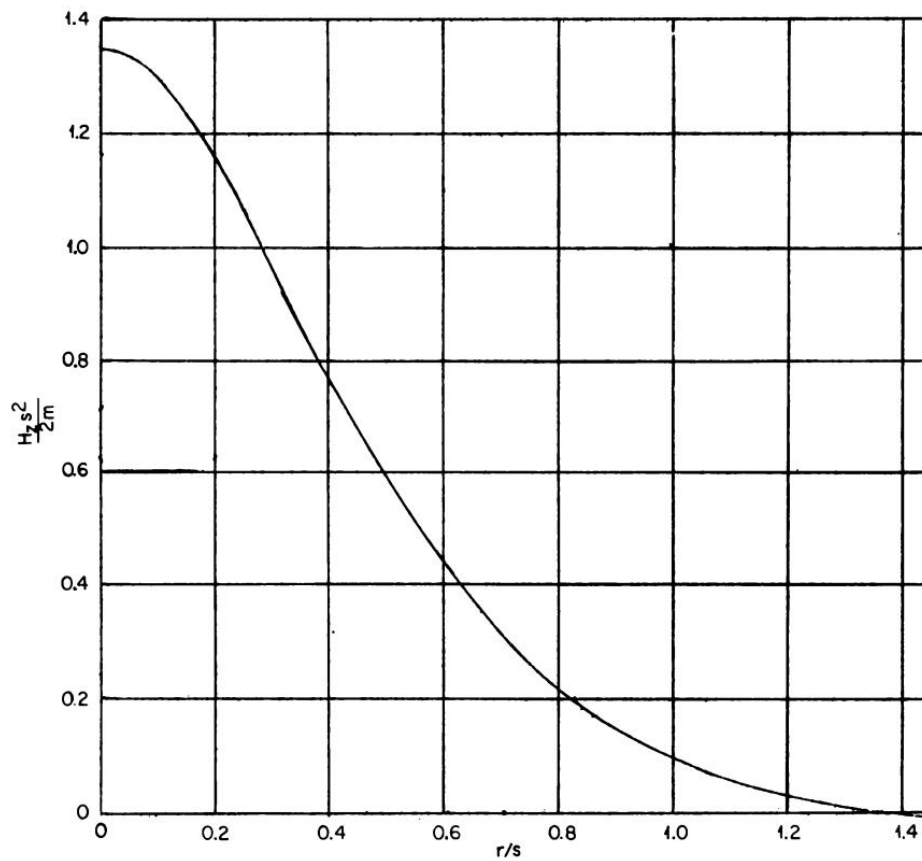


Fig. 1.4—Field H_z due to a cylindrical spike of height $h = 2s/7$ and area $A = 4\pi m/B_S$.

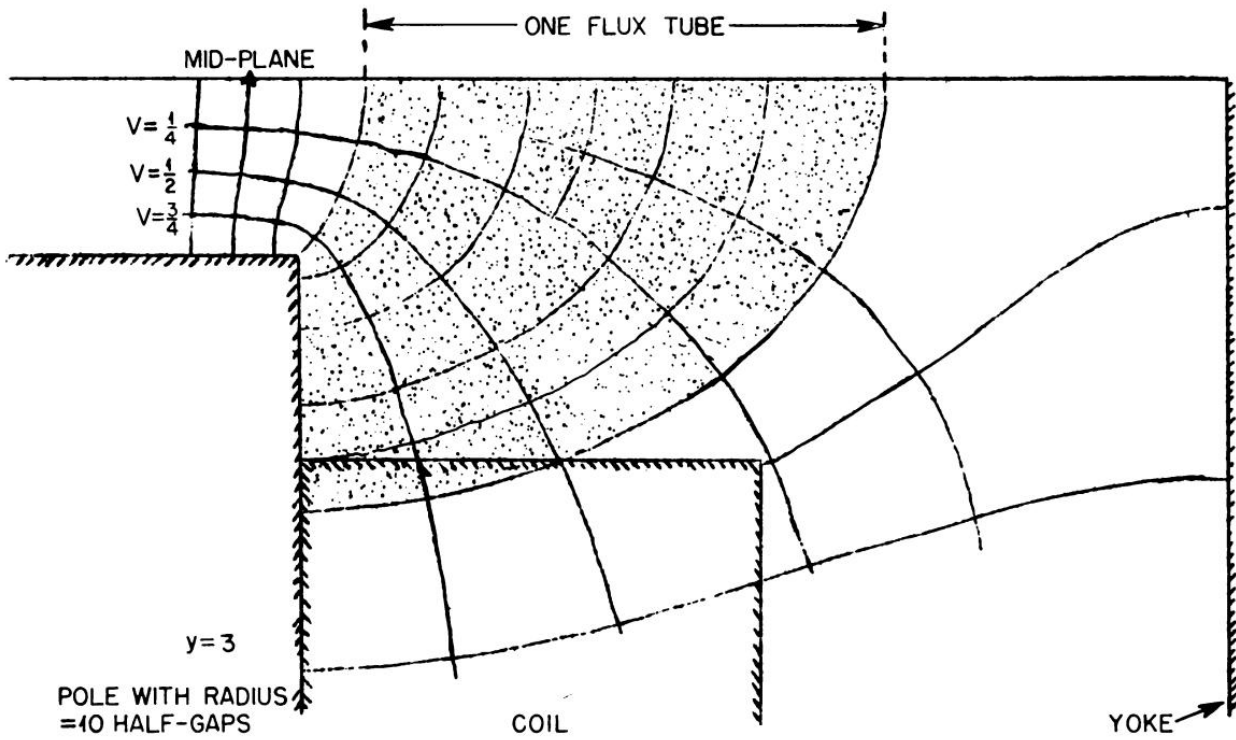


Fig. 1.5a—Three-dimensional flux map.

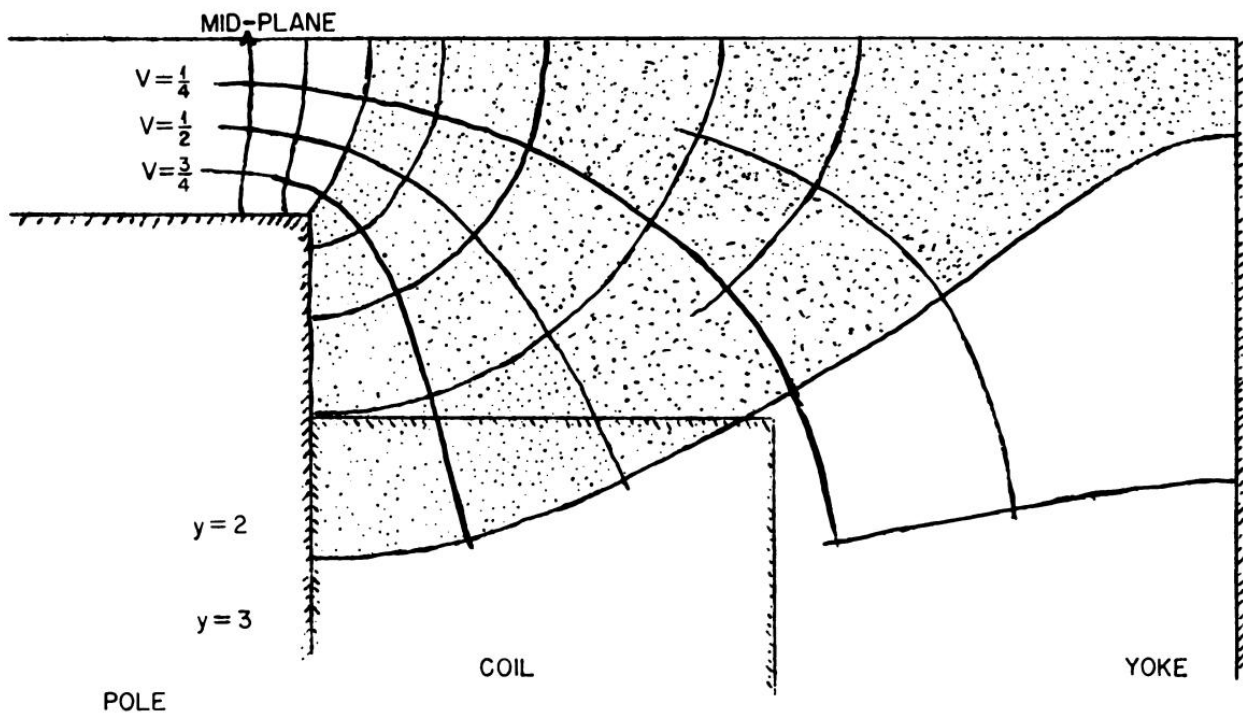


Fig. 1.5b—Two-dimensional flux map.

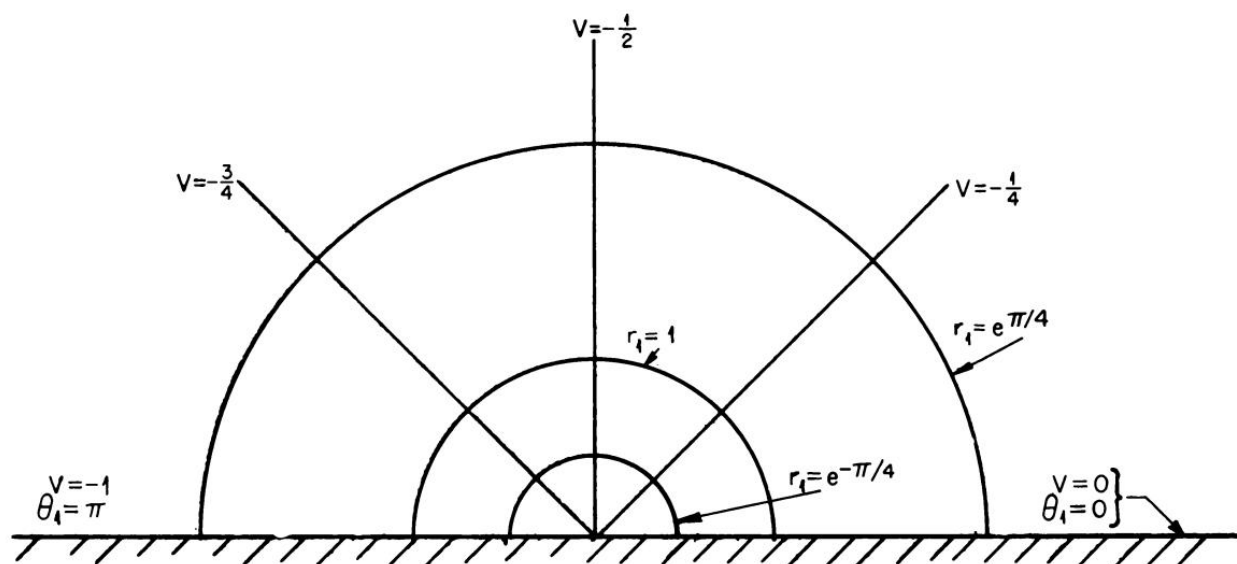


Fig. 1.6a—Field $V = -\theta_1/\pi$ mapped into the field of a square pole corner to give a square equipotential. z_1 plane.

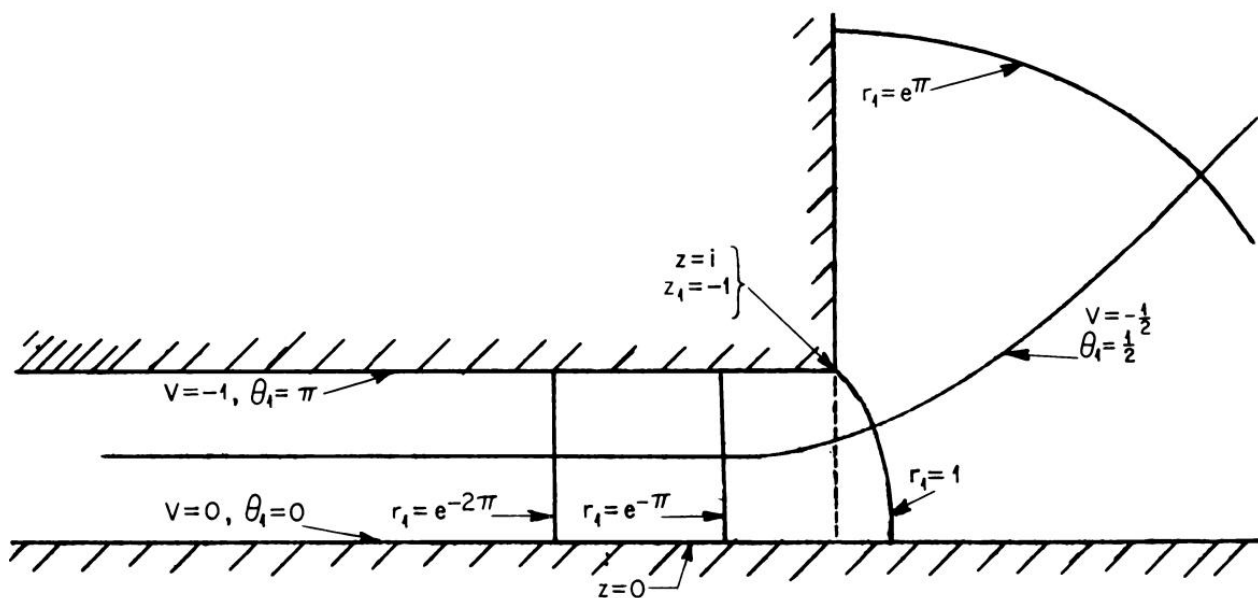


Fig. 1.6b—Field $V = -\theta_1/\pi$ mapped into the field of a square pole corner to give a square equipotential. z plane.

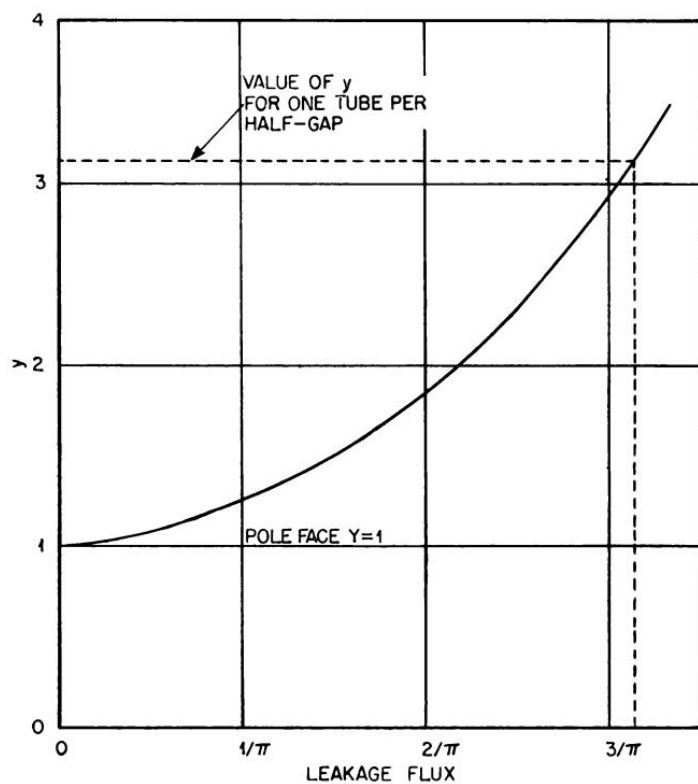


Fig. 1.7 — Leakage flux (into the square pole corner) per half-gap of circumference plotted against distance from mid-plane.

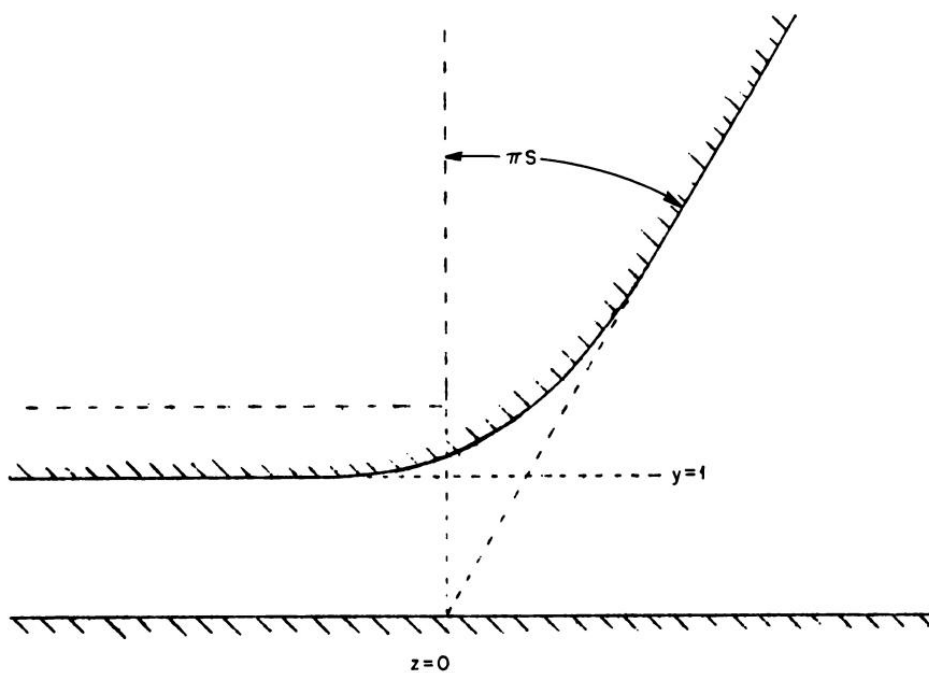


Fig. 1.8 — One of a set of equipotentials produced by the transformation which also produces the square equipotential.

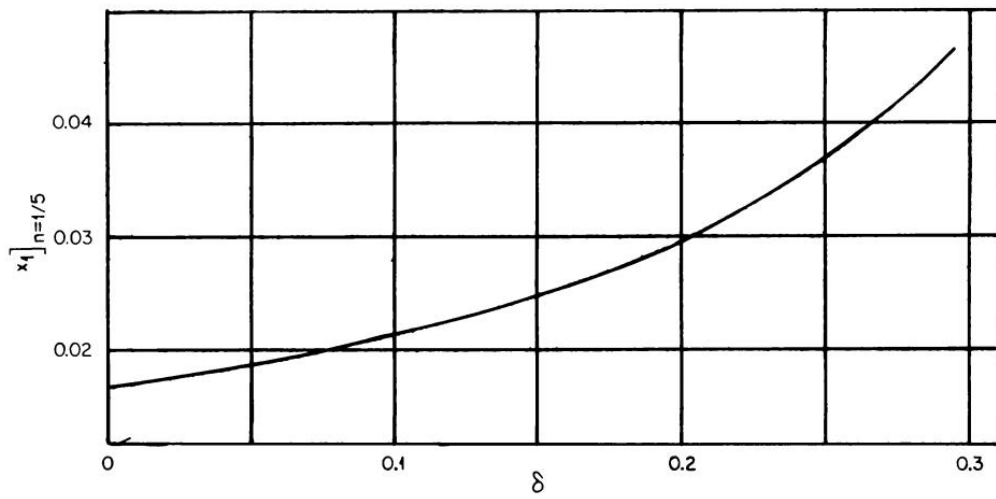


Fig. 1.9a— $x_1]_{n=1/5}$ as a function of δ with $m = 9$.

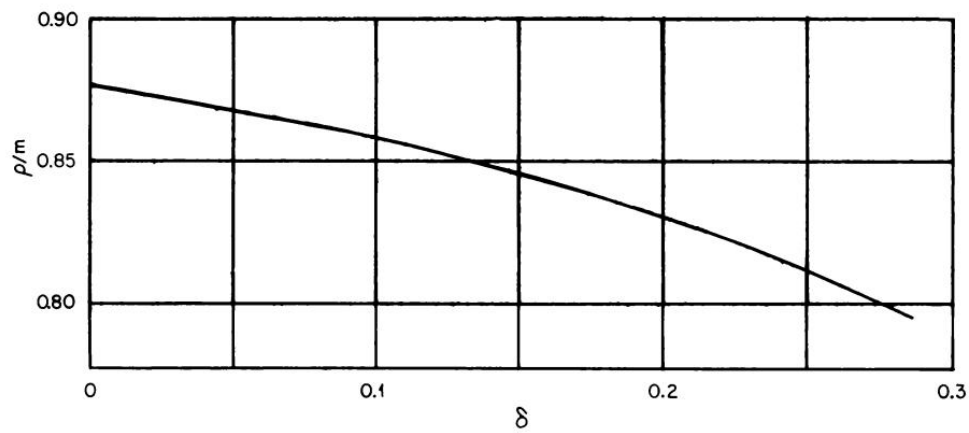


Fig. 1.9b— ρ/m as a function of δ with $m = 9$.

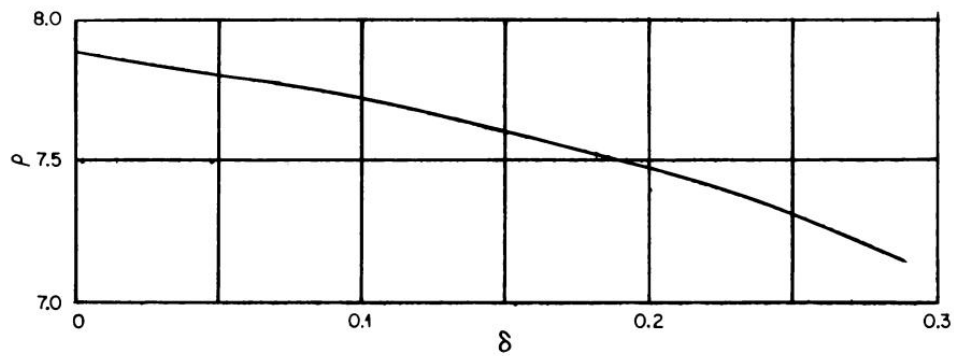


Fig. 1.9c— ρ as a function of δ with $m = 9$.

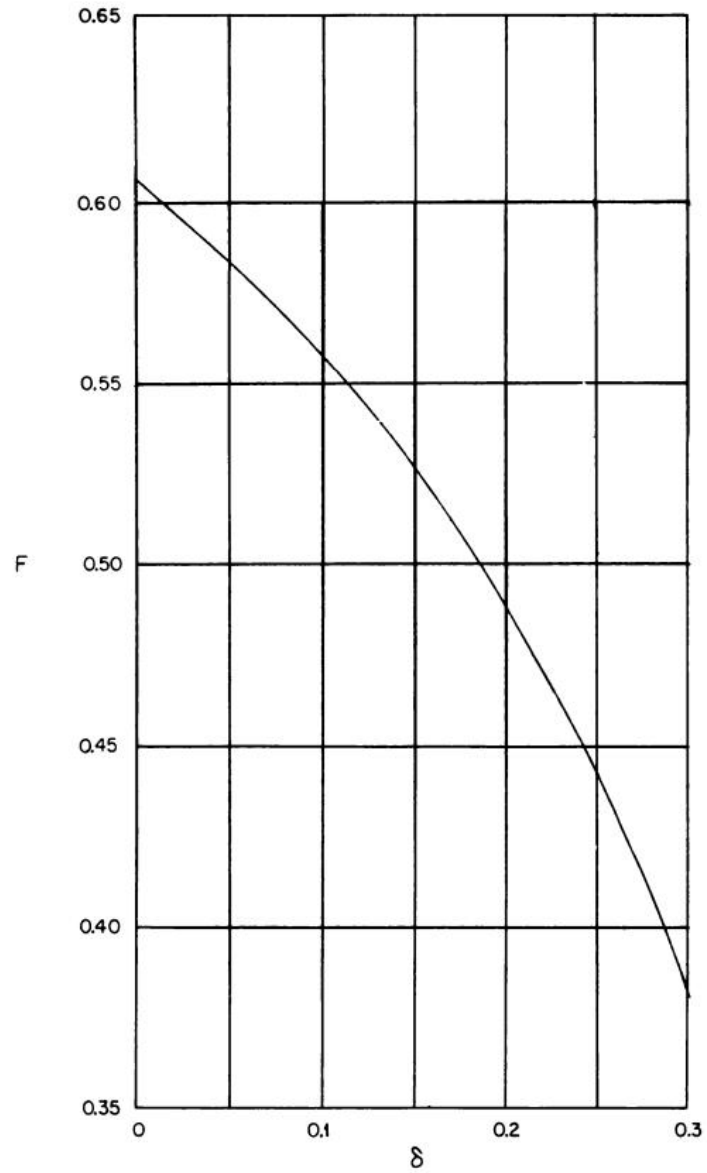


Fig. 1.10—Flux efficiency F vs. δ with $m = 9$.

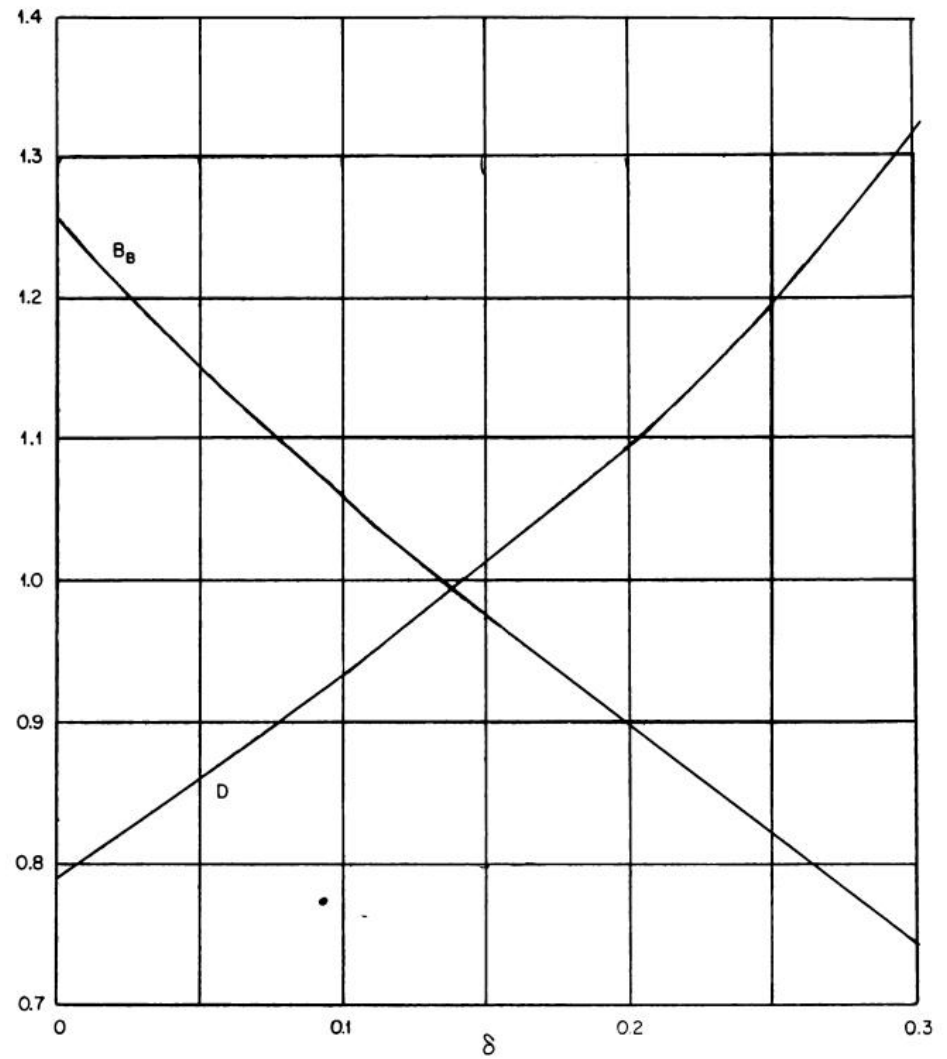


Fig. 1.11—Flux density at $y = 3$, B_B , and the relative flux density, $D = H_0/B_B$, are plotted as functions of δ .

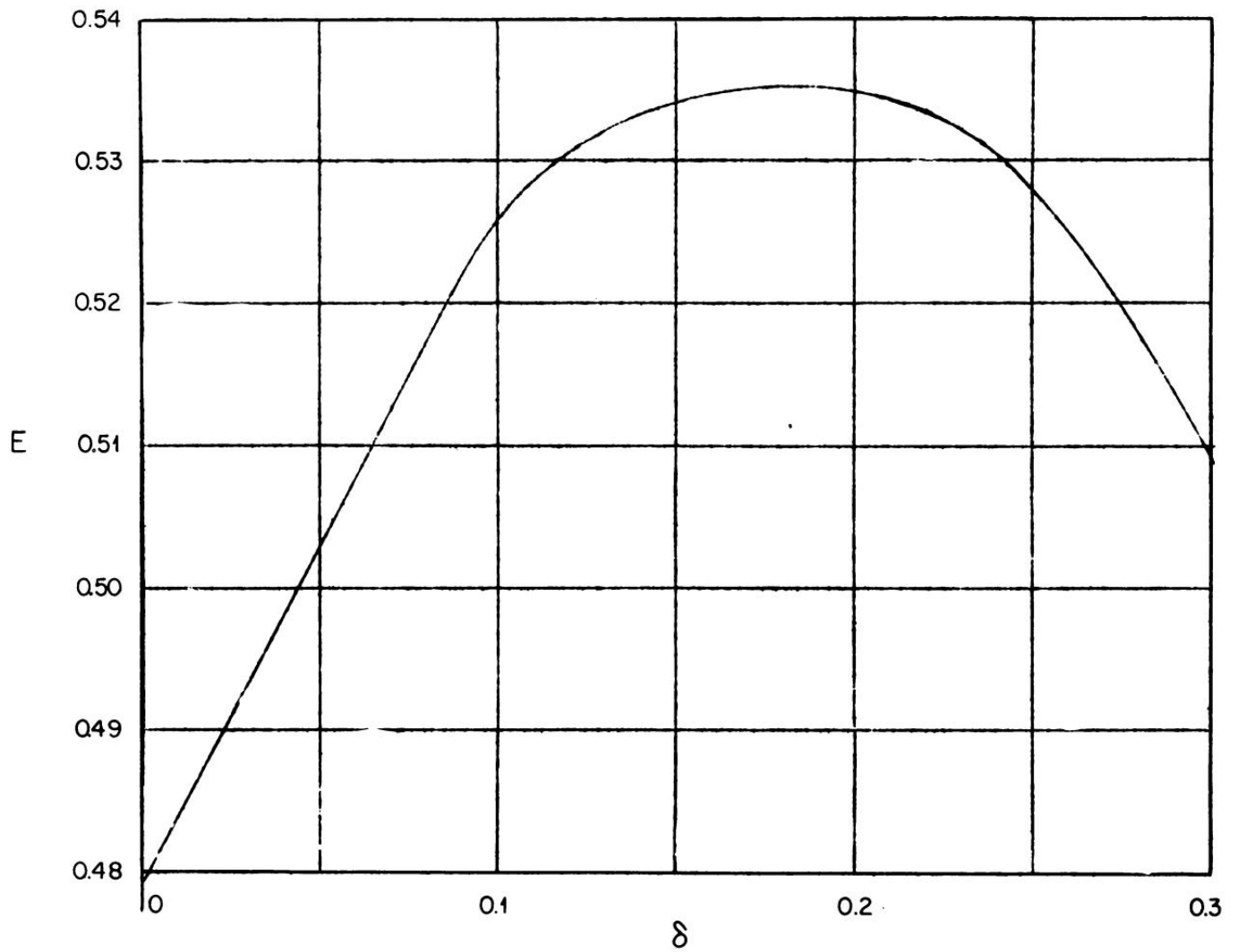


Fig. 1.12a—Pole-tip efficiency E plotted as a function of δ .

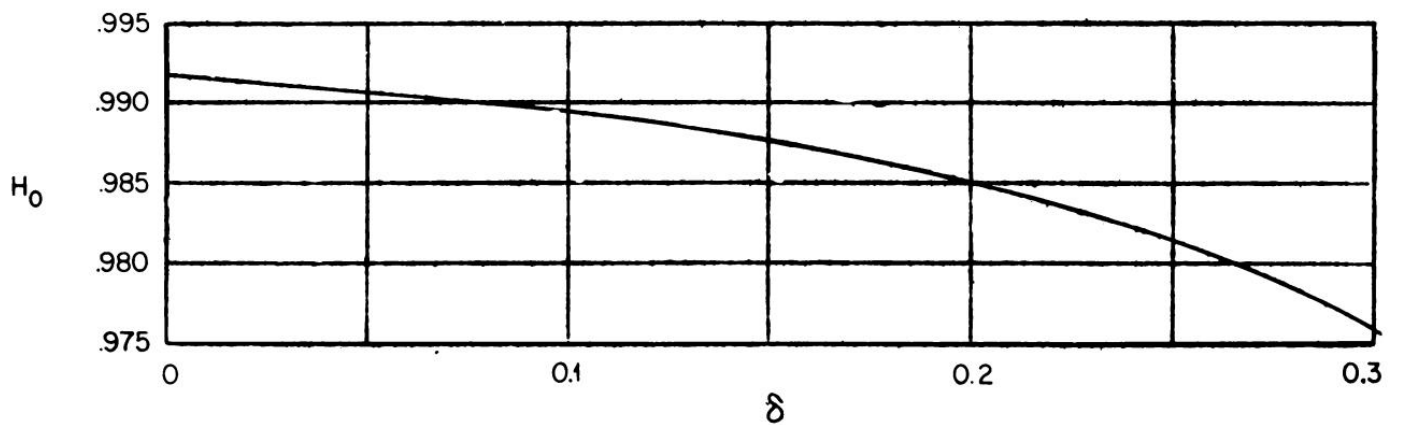


Fig. 1.12b—Field strength H_0 , where $n = 1/5$, plotted as a function of δ .

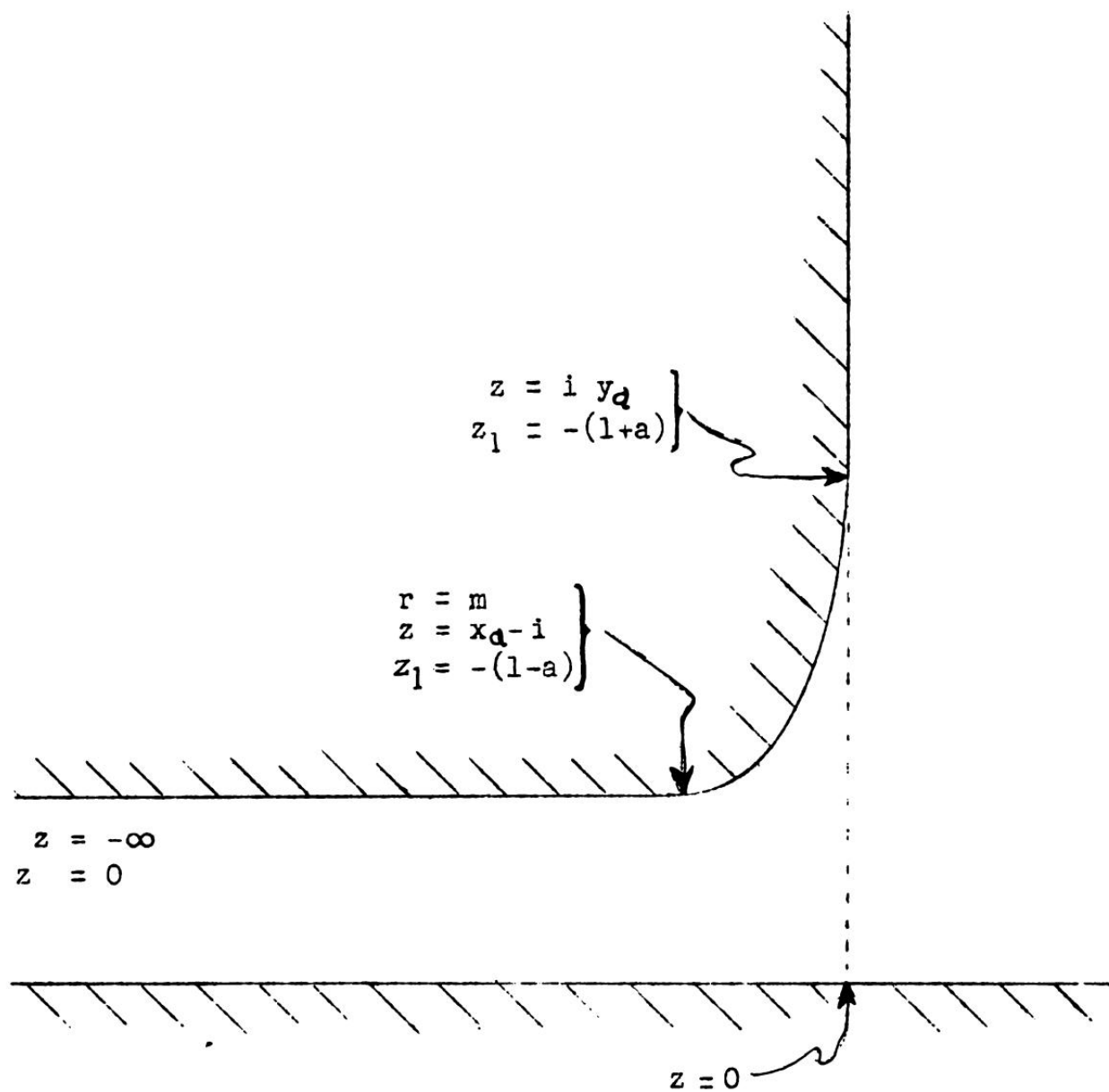
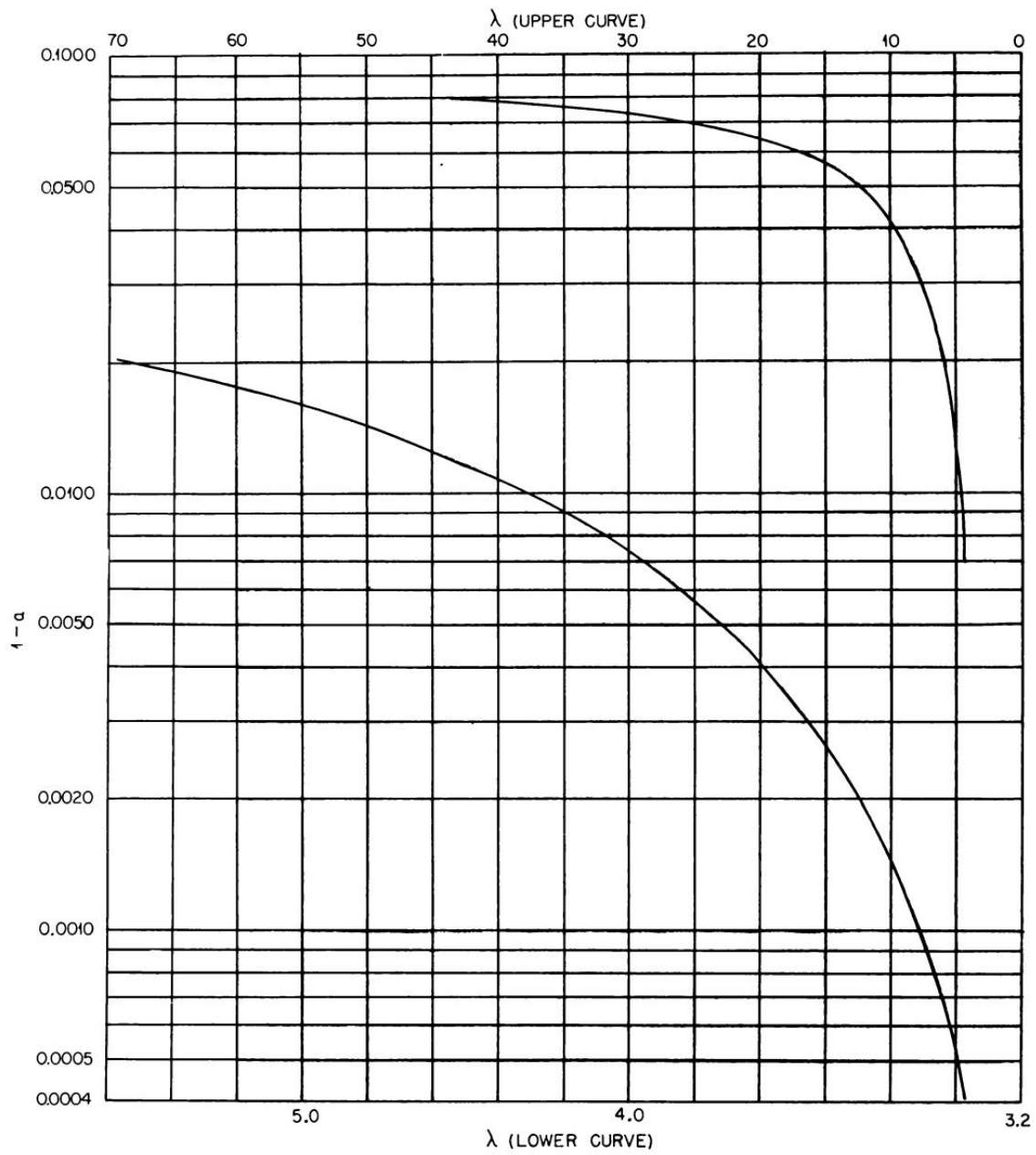


Fig. 1.13—Rounded equipotential, illustrating the z plane of the transformation given by Eq. 36.

Fig. 1.14— λ vs. $1-a$.

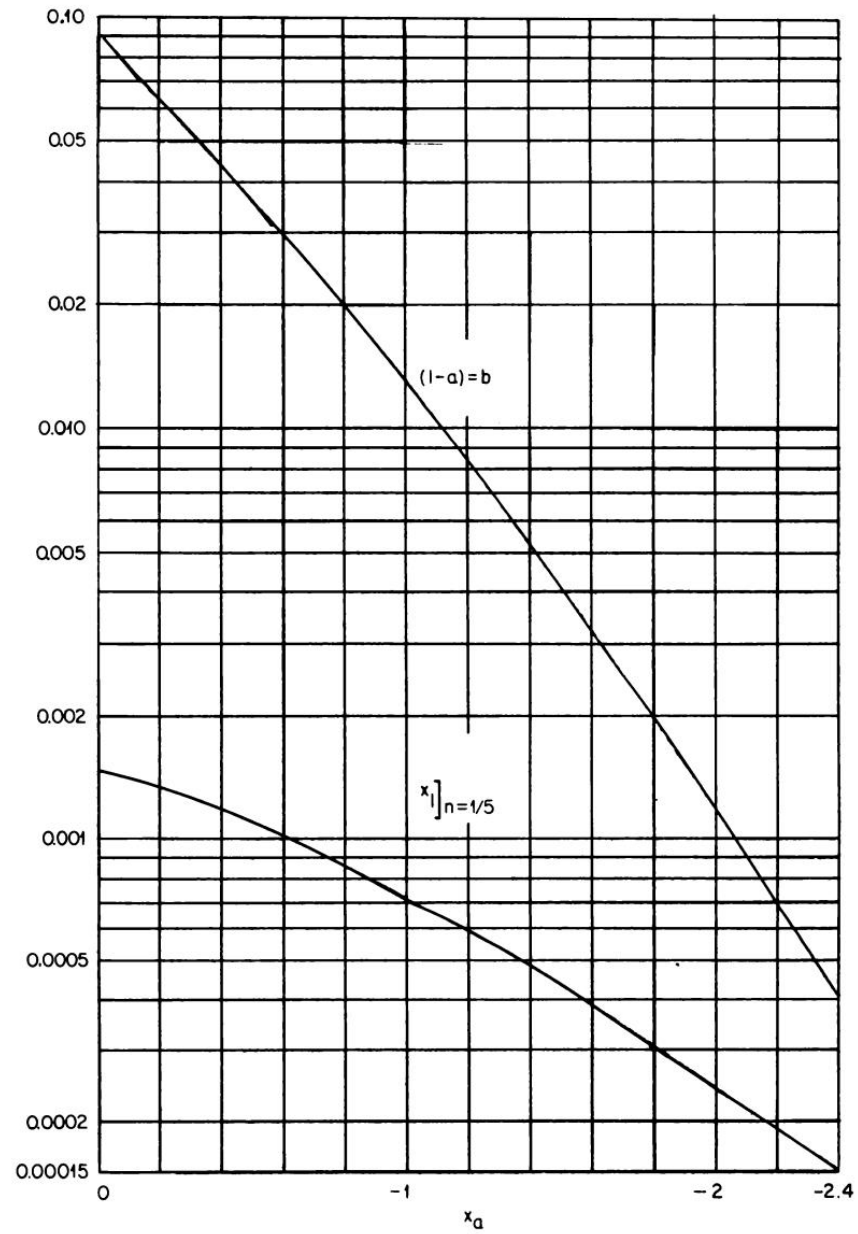


Fig. 1.15— $1-a$ vs. x_α and $x_1]_{n=1/5}$ vs. x_α .

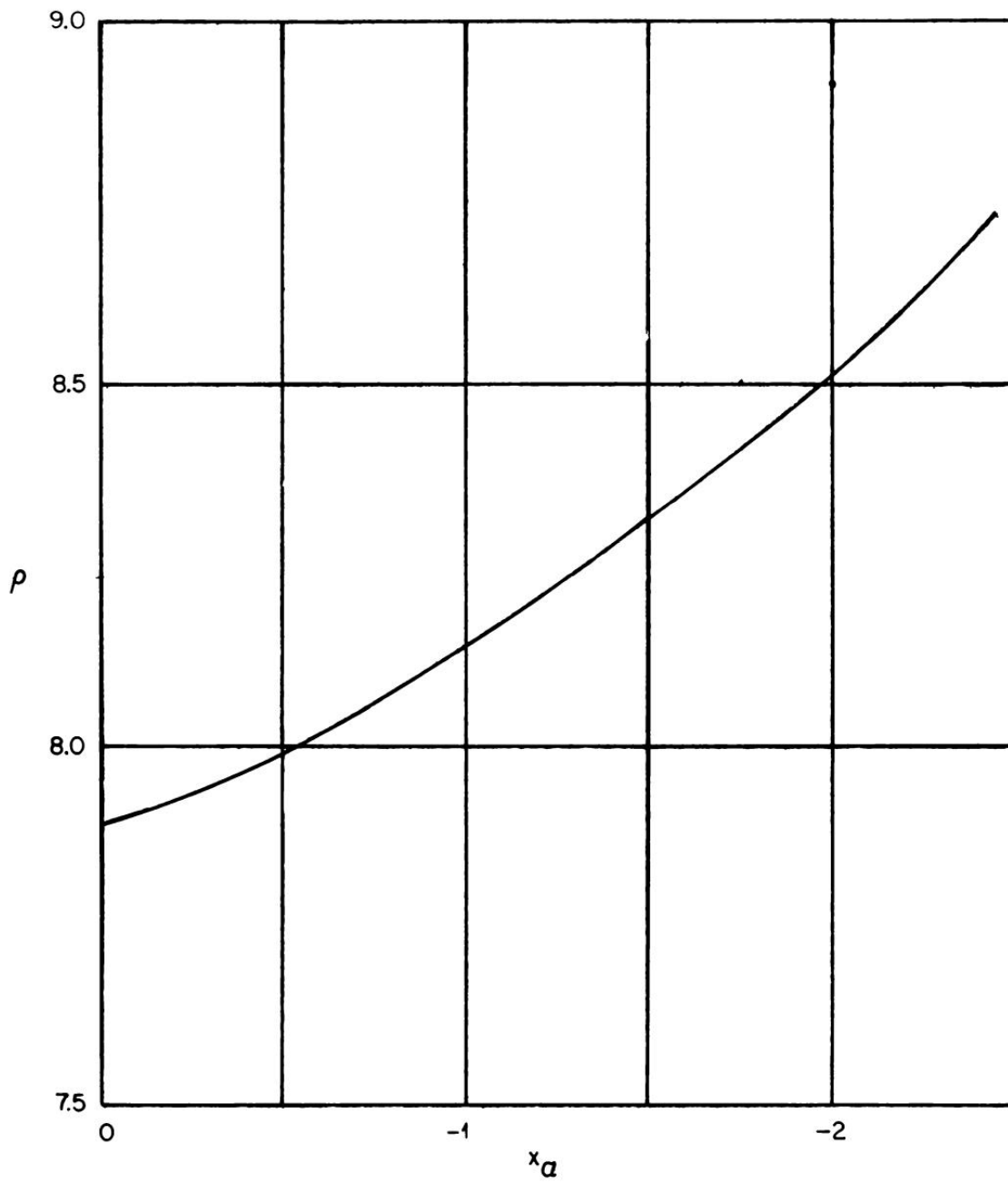
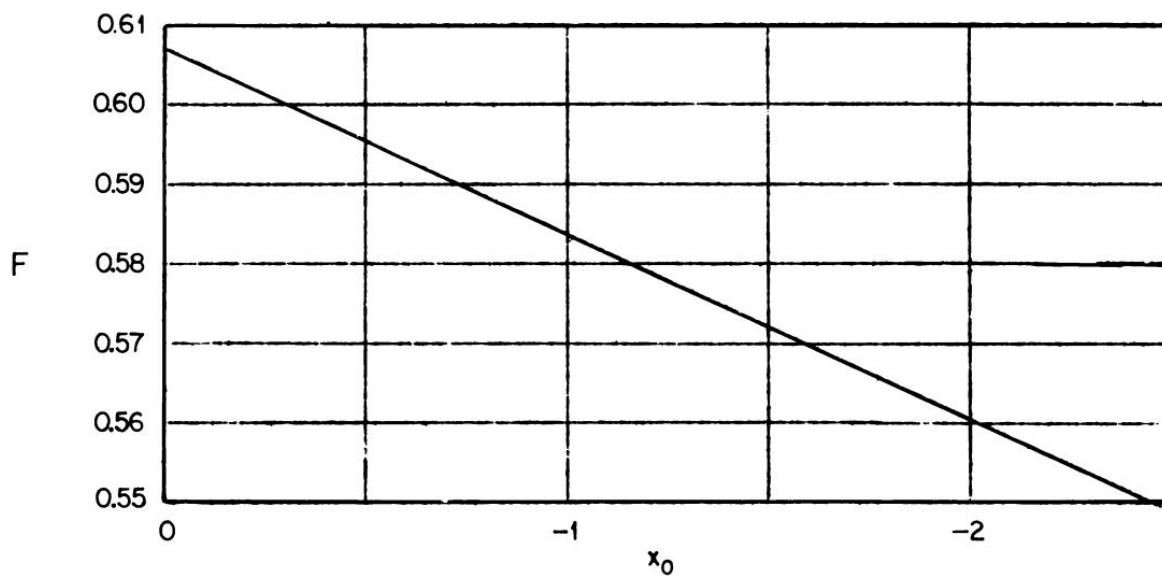
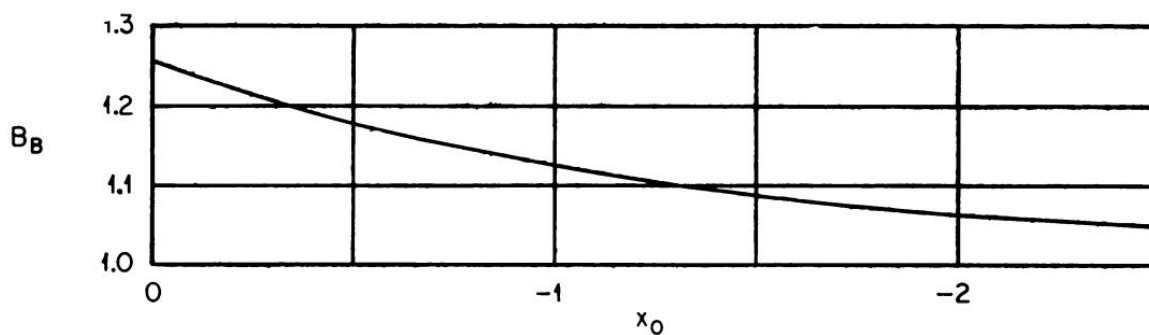
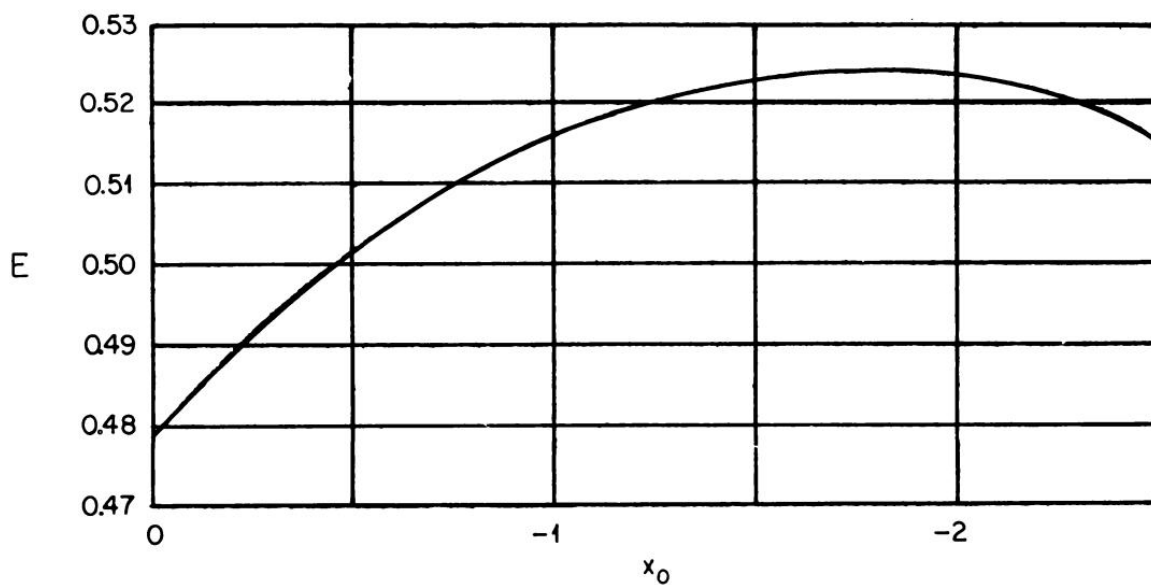
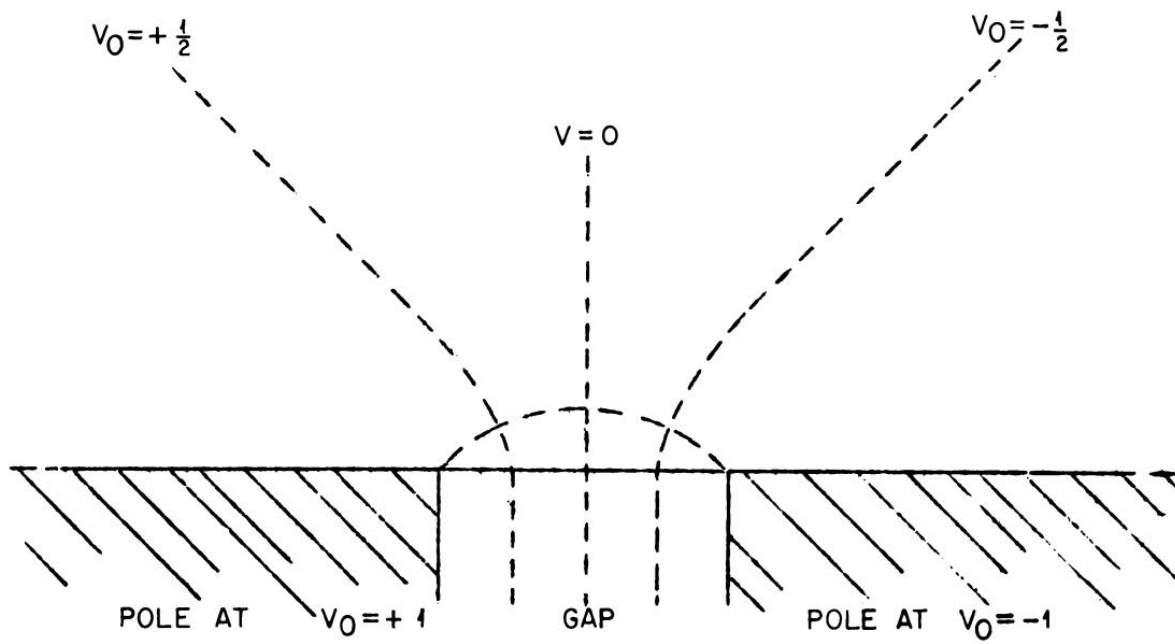
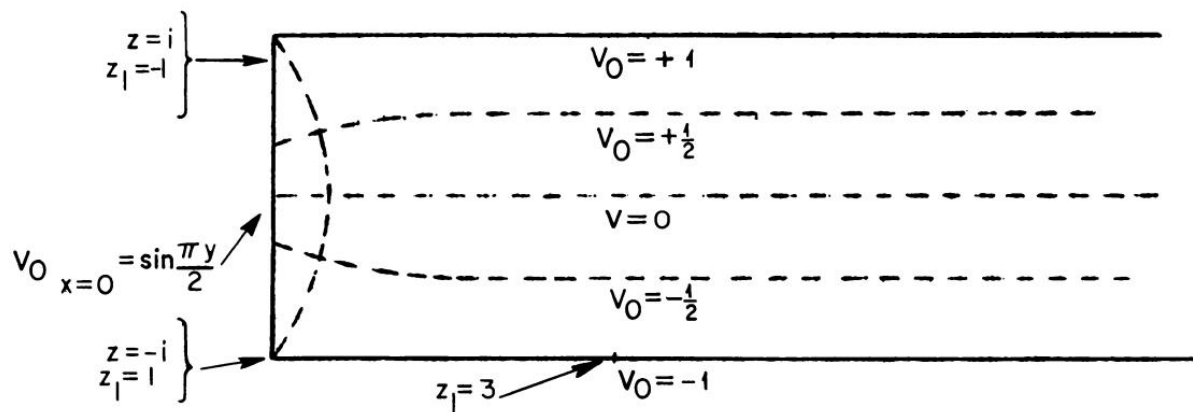


Fig. 1.16—Useful radius ρ , for the fields produced by the rounded pole corners, is given as a function of x_α .

Fig. 1.17a—Performance of rounded pole tip; F , flux efficiency.Fig. 1.17b—Performance of rounded pole tip; B_B , flux density in pole base.Fig. 1.17c—Performance of rounded pole tip; E , pole-tip efficiency.

Fig. 1.18a—Shim that crosses the gap; z_1 plane.Fig. 1.18b—Shim that crosses the gap; z plane.

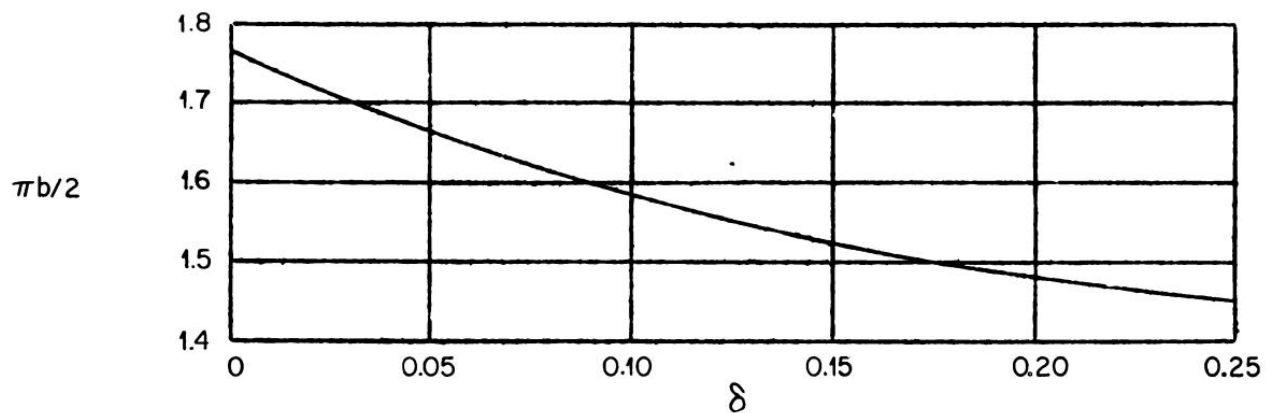


Fig. 1.19a— $\pi b/2$ given as a function of δ for a shim that crosses the gap.

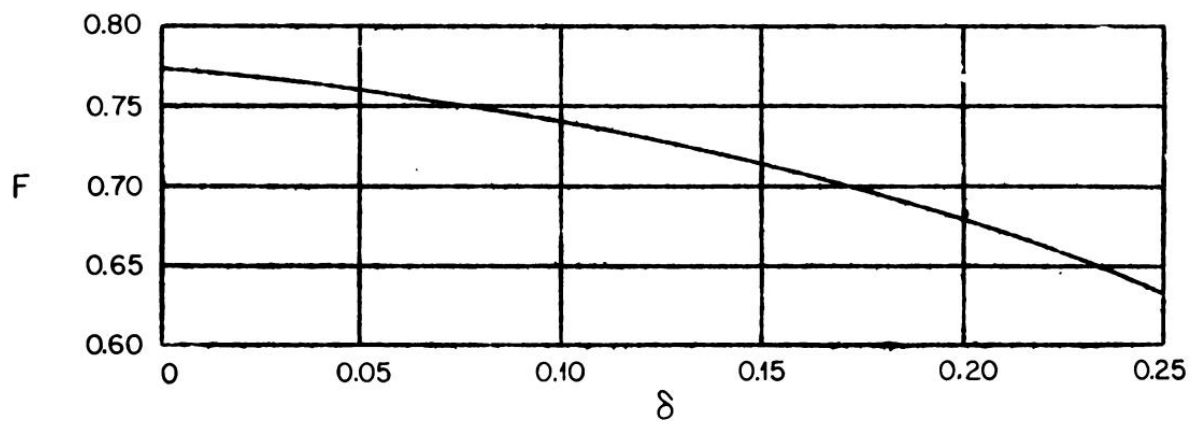


Fig. 1.19b— F given as a function of δ for a shim that crosses the gap.

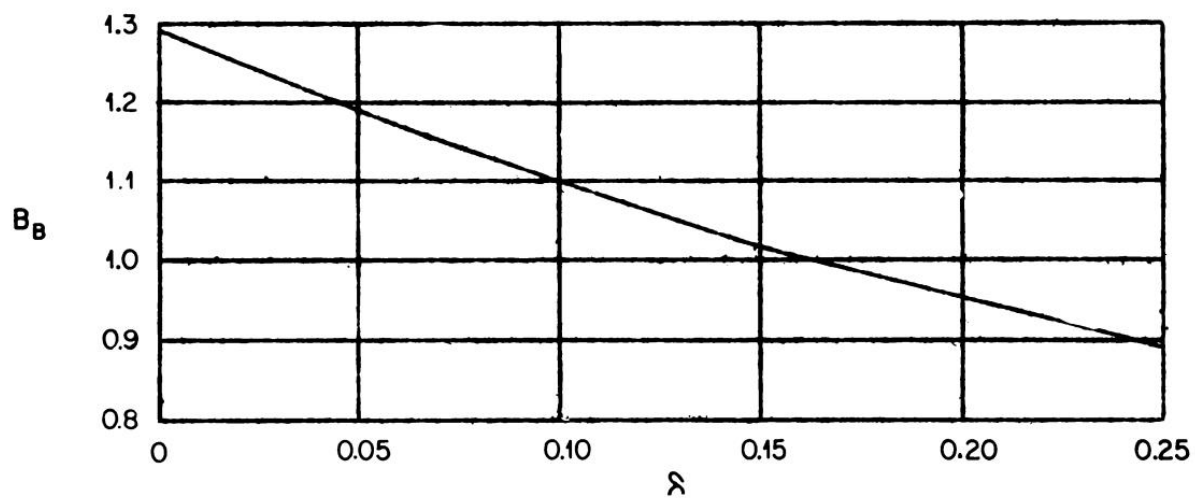


Fig. 1.19c— B_B given as a function of δ for a shim that crosses the gap.

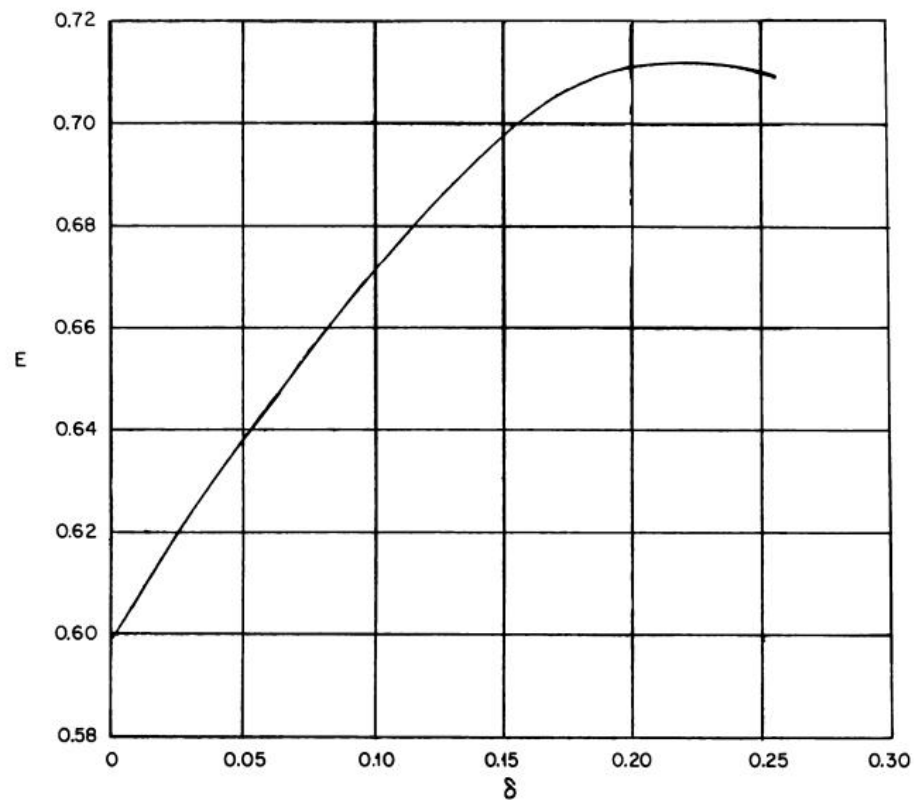


Fig. 1.20—Pole-tip efficiency E obtained with a shim crossing the edge of the gap is shown as a function of δ , where δ is the angle between the side of the pole and the axis of the magnet; $m = 9$.

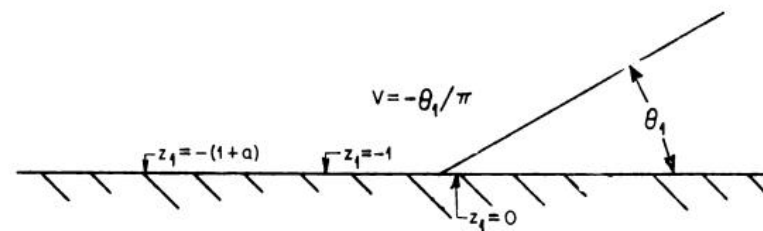


Fig. 1.21a—Pole with a raised edge; z_1 plane.

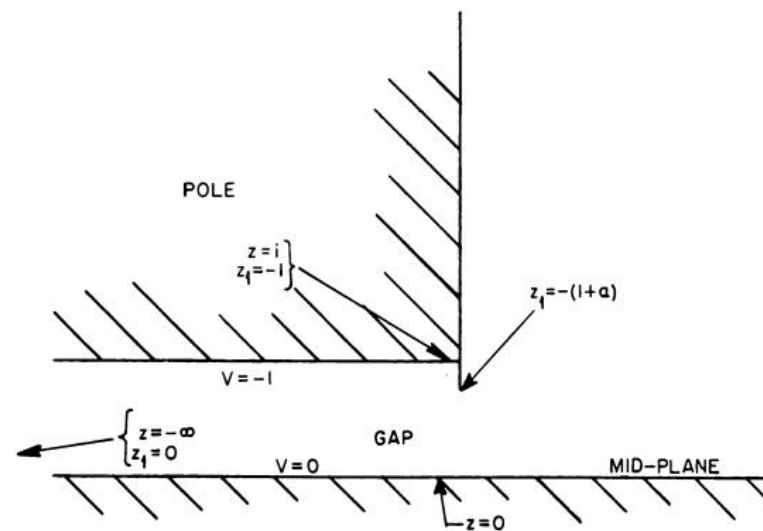


Fig. 1.21b—Pole with a raised edge; z plane.

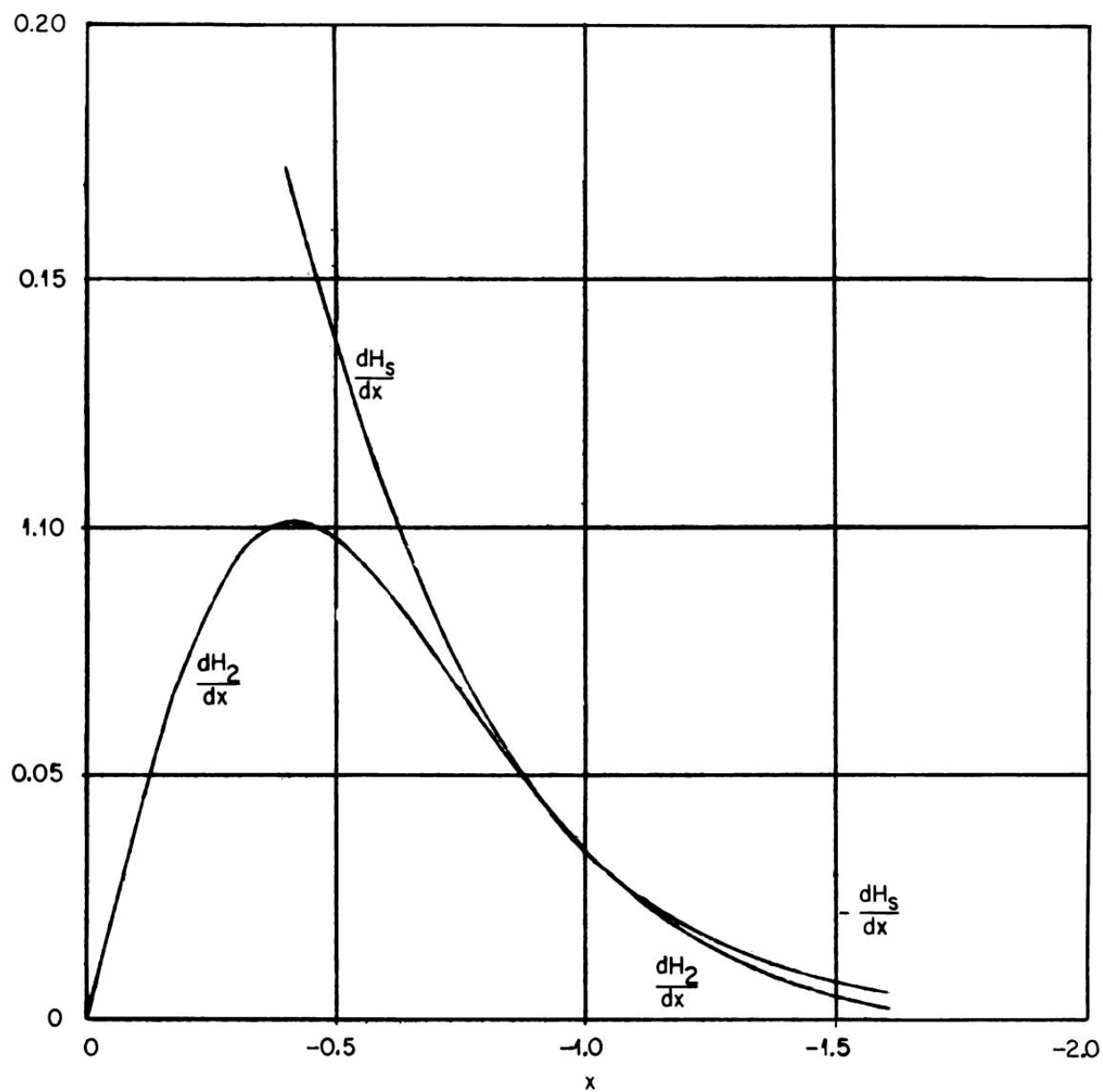


Fig. 1.22—Strength of the dipole at $x = 0$, $y = 1$, which will produce a monotone decreasing field with the greatest useful radius, determined by making the derivative of the dipole field equal and opposite to that of the square equipotential at one point.

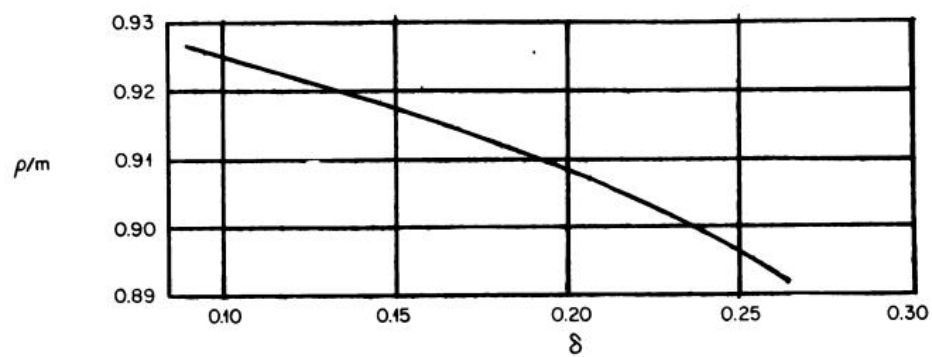


Fig. 1.23a—Square equipotential plus dipole; ρ/m vs. δ ; $m = 9$.

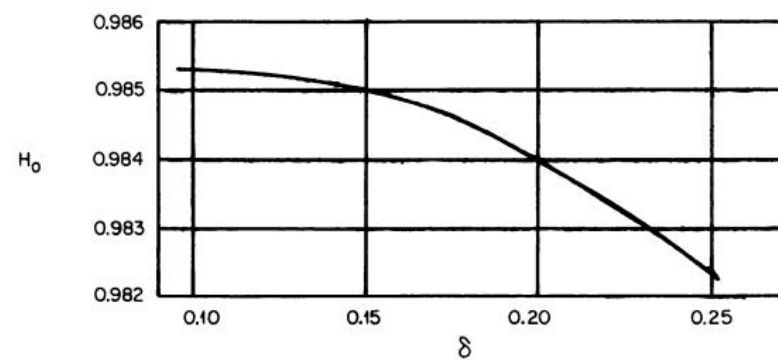


Fig. 1.23c—Square equipotential plus dipole; H_0 vs. δ ; $m = 9$.

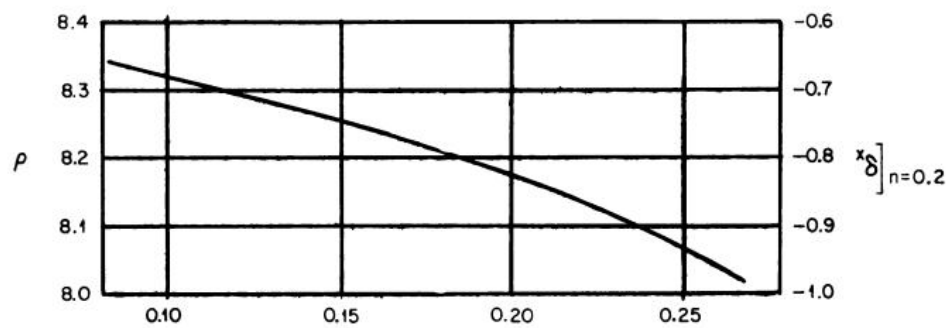


Fig. 1.23b—Square equipotential plus dipole; ρ vs. δ ; $m = 9$.

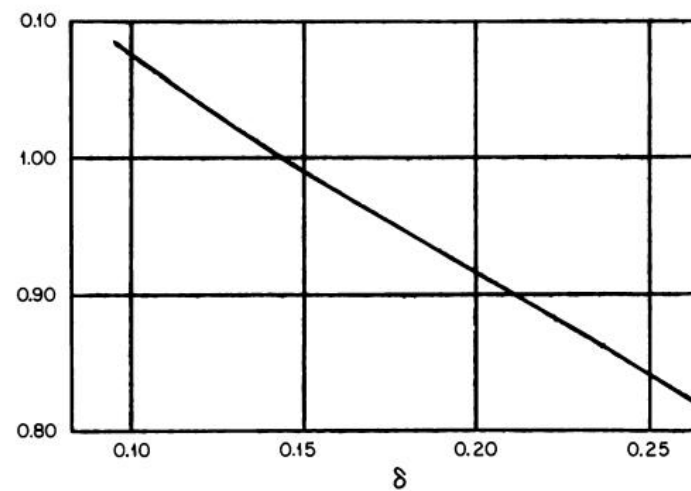


Fig. 1.23d—Square equipotential plus dipole; B_B vs. δ ; $m = 9$.

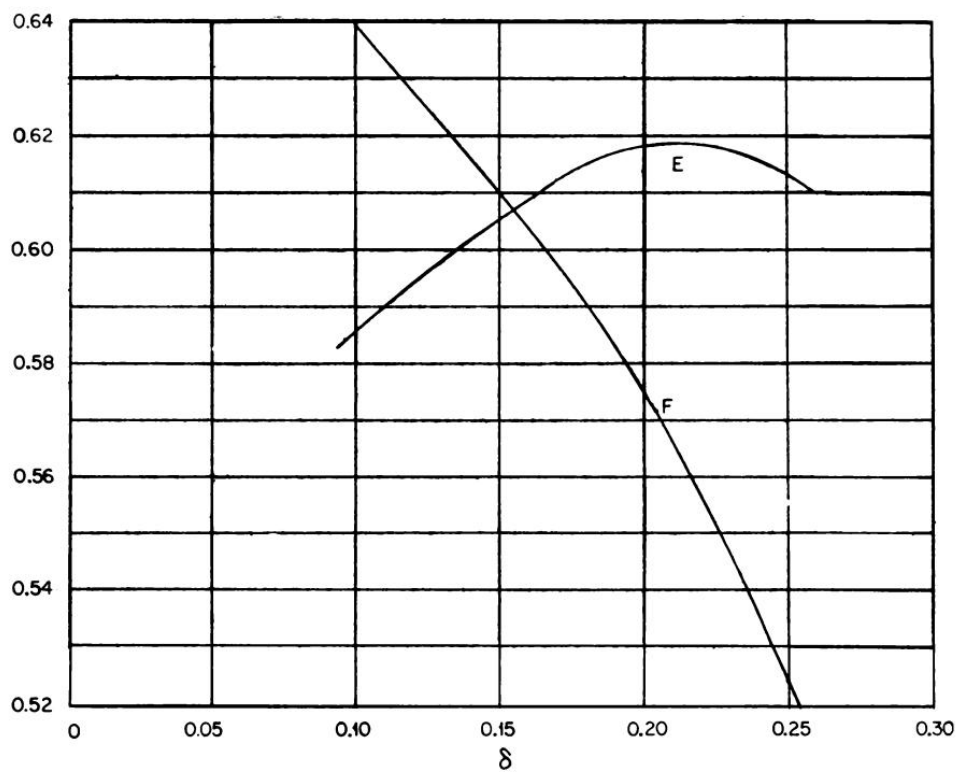


Fig. 1.24a— Properties of a pole with a dipole field added. F, flux efficiency; E, pole-tip efficiency.

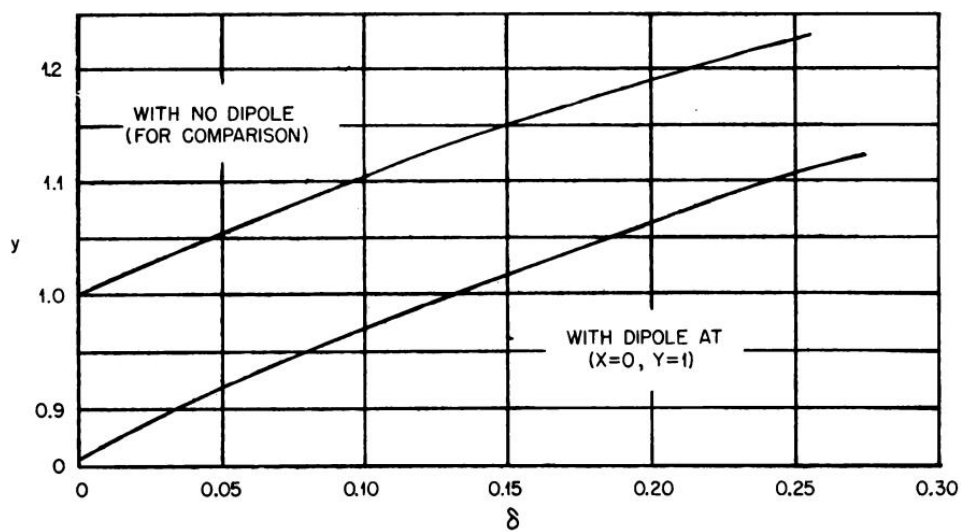


Fig. 1.24b— Distance from mid-plane to point where $V = 1$ as a function of δ .

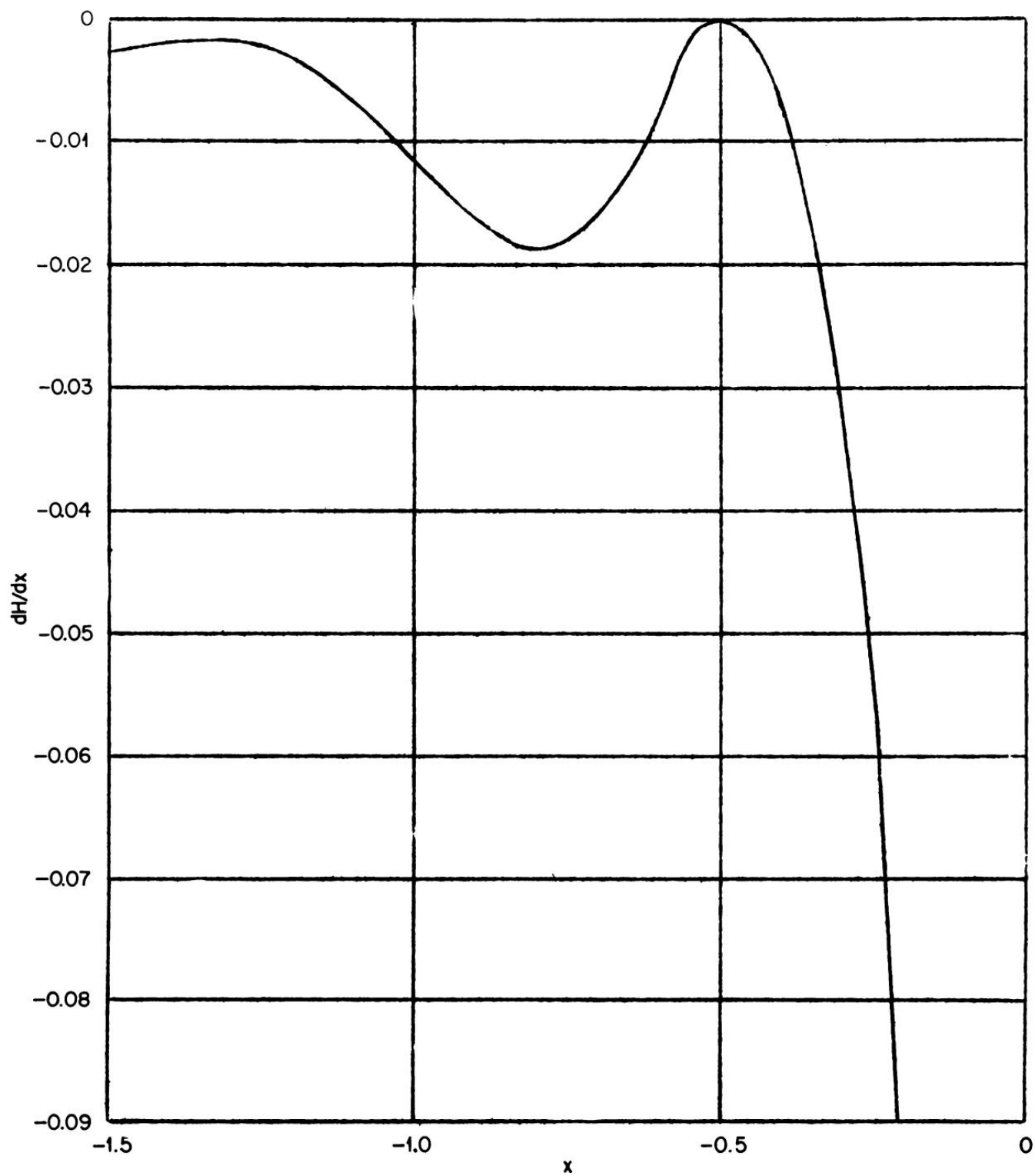


Fig. 1.25— dH/dx vs. x ; dH/dx for the field due to a square equipotential plus a quadrupole of strength $k = 0.0900$ at $x = 0$, $y = 1$, plus a dipole of strength $k = 0.0218$ at $x = -0.8$, $y = 1$.

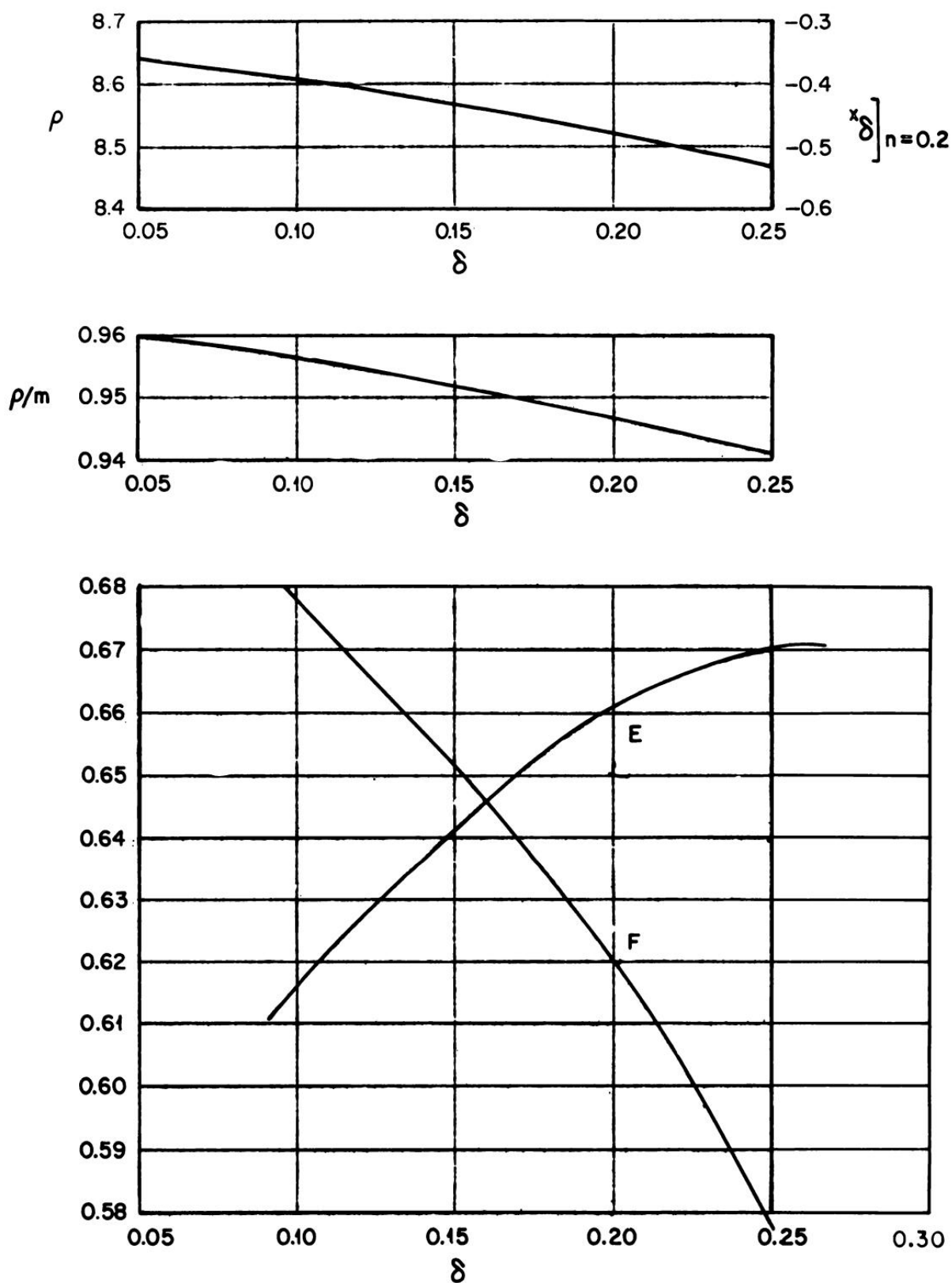


Fig. 1.26—Pole tip with quadrupole added; properties of a square pole tip with a quadrupole of strength 0.090 at $x = 0$, $y = 1$, and a dipole of strength 0.218 at $x = -0.8$, $y = 1$.

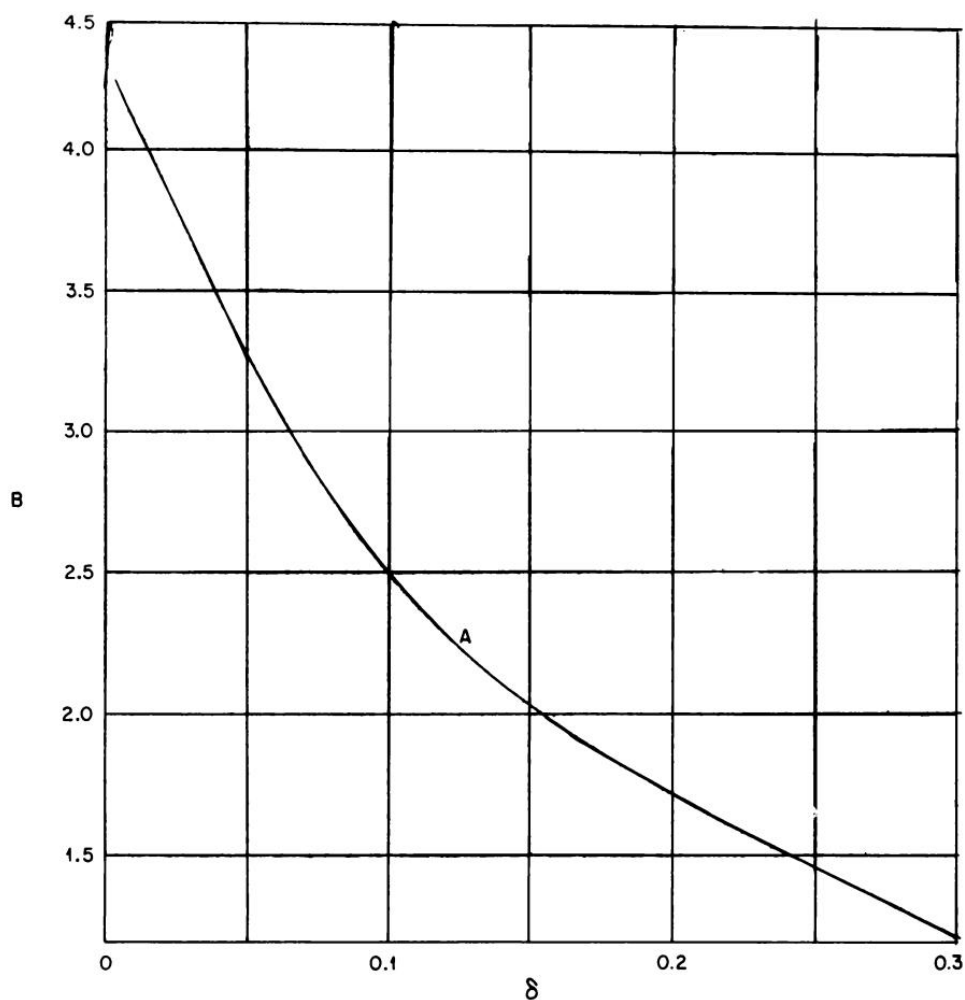


Fig. 1.27a—Further properties of a square pole with a quadrupole at $x = 0$, $y = 1$. Curve A, approximate flux density at $V = 1$, $x = 0$, i.e., at the tip of the raised edge of the shim, plotted as a function of δ .

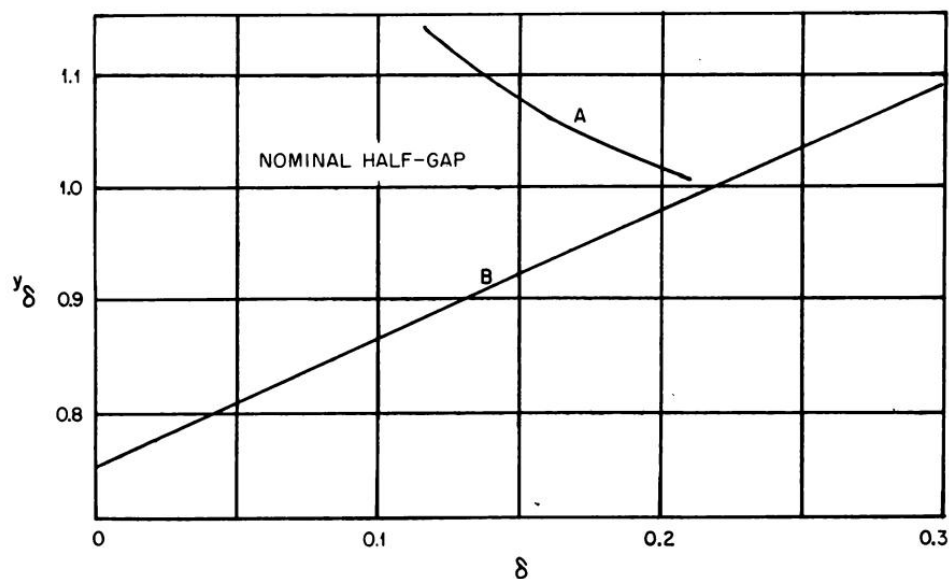


Fig. 1.27b—Further properties of a square pole with a quadrupole at $x = 0$, $y = 1$. Curve A, distance of $V = 1$ from mid-plane at $x = -0.4$ or $x_\delta = -0.4/(1 - 2\delta)$. Curve B, distance of $V = 1$ from mid-plane at the pole edge, $x = 0$.

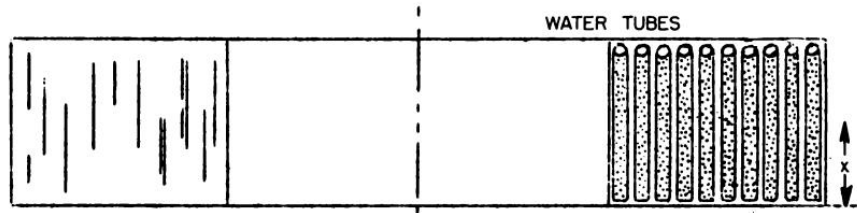


Fig. 1.28a—Cooling arrangement with water tubes on top of conductors, conventional winding.

$$T - T_0 = \frac{i^2 \rho}{8f^2 w^2 k} \left(1 - \frac{x^2}{h^2} \right)$$

$$T_m - T_0 = \frac{i^2 \rho}{8f^2 w^2 k}$$

T = temperature of conductor

T_0 = temperature of water

k = thermal conductivity, $\frac{\text{watts}}{^\circ\text{C cm}}$

f = space factor

T_m = maximum conductor temperature

i = ampere turns for magnet

ρ = resistivity of conductor, ohm cm

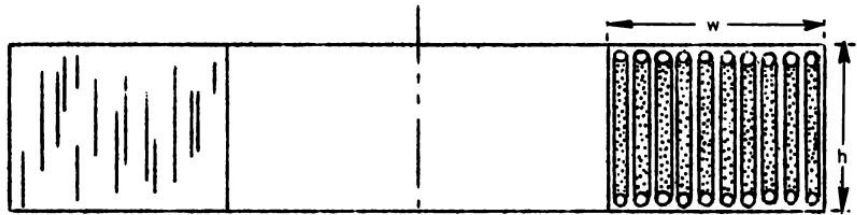


Fig. 1.28b—Cooling arrangement with water tubes on both top and bottom of conductors, conventional winding.

$$T - T_0 = \frac{i^2 \rho}{8f^2 w^2 k} \left(\frac{x}{h} - \frac{x^2}{h^2} \right)$$

$$T_m - T_0 = \frac{i^2 \rho}{32f^2 w^2 k}$$

T = temperature of conductor

T_0 = temperature of water

k = thermal conductivity, $\frac{\text{watts}}{^\circ\text{C cm}}$

f = space factor

T_m = maximum conductor temperature

i = ampere turns for magnet

ρ = resistivity of conductor, ohm cm

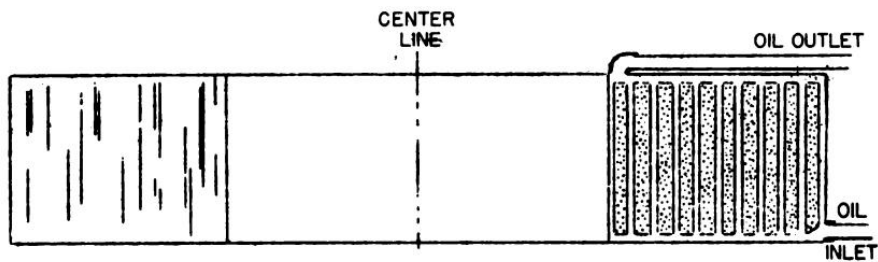


Fig. 1.28c—Cooling arrangement with oil flowing between the conductors, conventional winding.

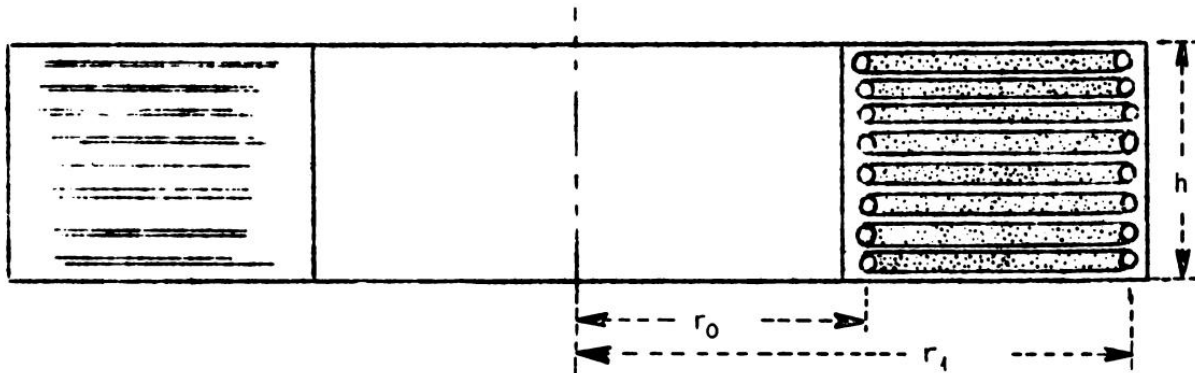


Fig. 1.29a—Cooling arrangement with water tubes inside and outside the conductors, helical winding.

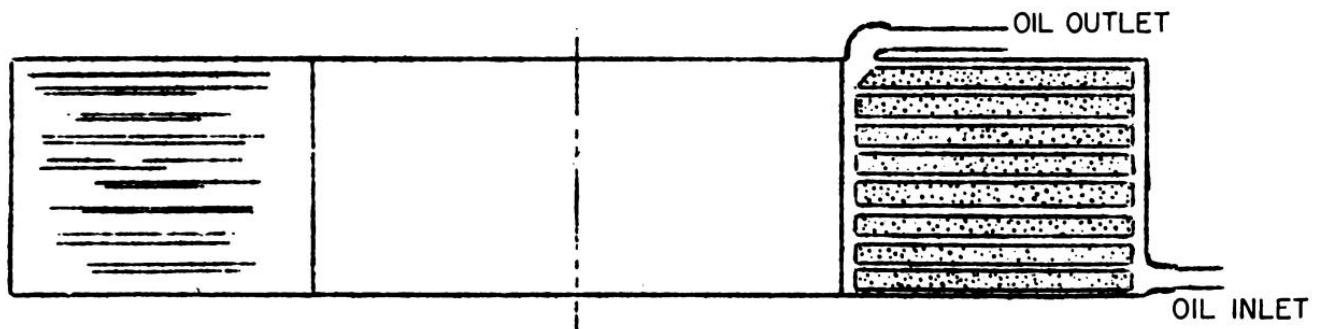


Fig. 1.29b—Cooling arrangement with oil flowing between the conductors, helical winding.

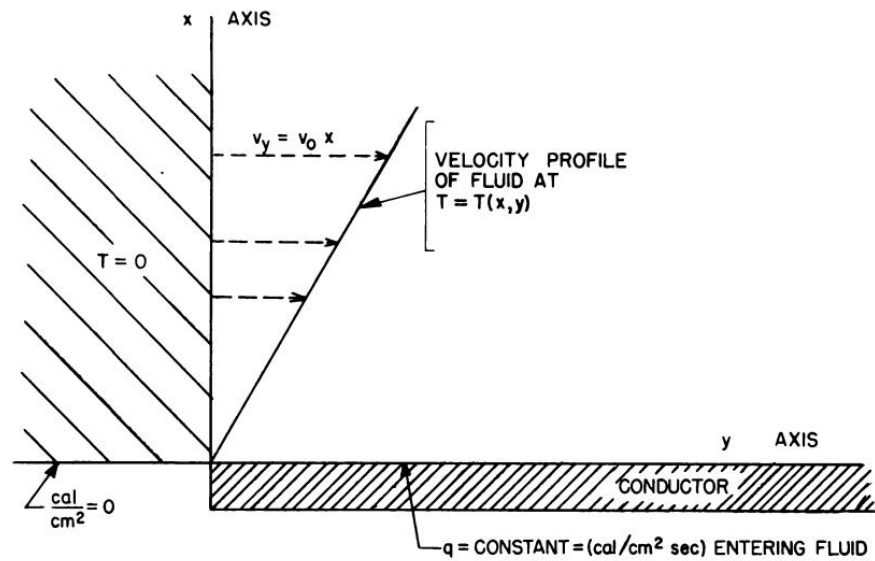


Fig. 1.30—Oil cooling, coordinates and definitions. q , heat delivered to fluid at $x = 0, y > 0$, $\text{cal}/\text{cm}^2/\text{sec}$; σ , specific heat per unit volume of fluid, $\text{cal}/^\circ\text{C}/\text{cm}^3/\text{sec}$; k , thermal conductivity of fluid, $\text{cal}/^\circ\text{C}/\text{cm}/\text{sec}$.

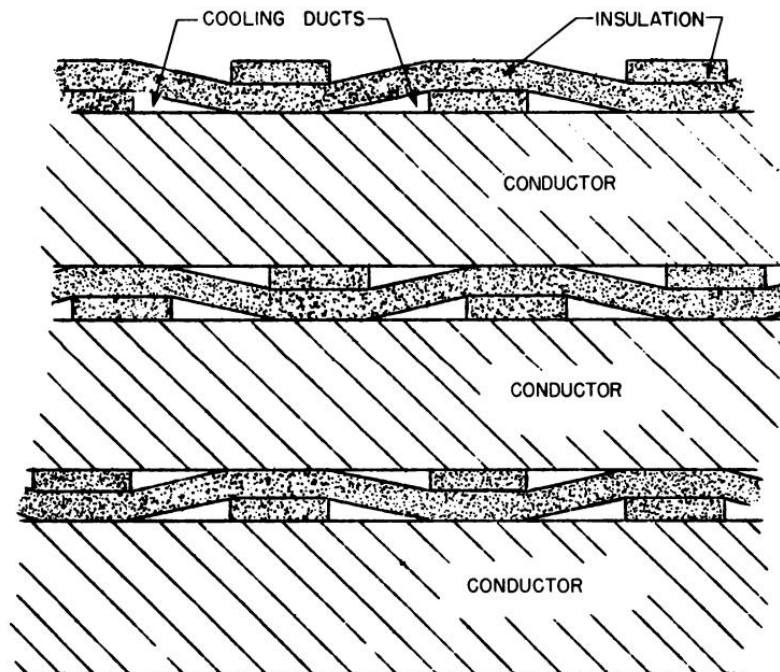


Fig. 1.31—Cooling and insulating arrangement proposed by Westinghouse; coolant flows perpendicular to paper through triangular ducts. (Not to scale).

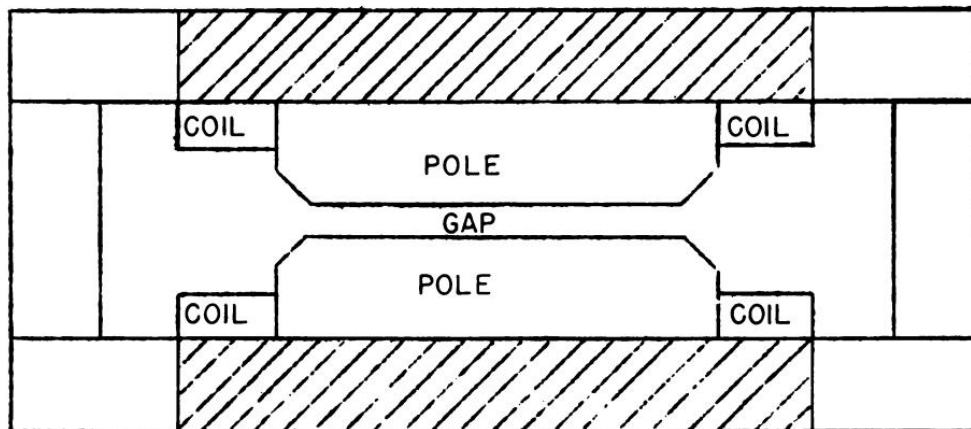


Fig. 1.32—Dependence of steel on coil outside diameter; shaded area represents the part of the steel whose cost is a function of the outside diameter of the coil.

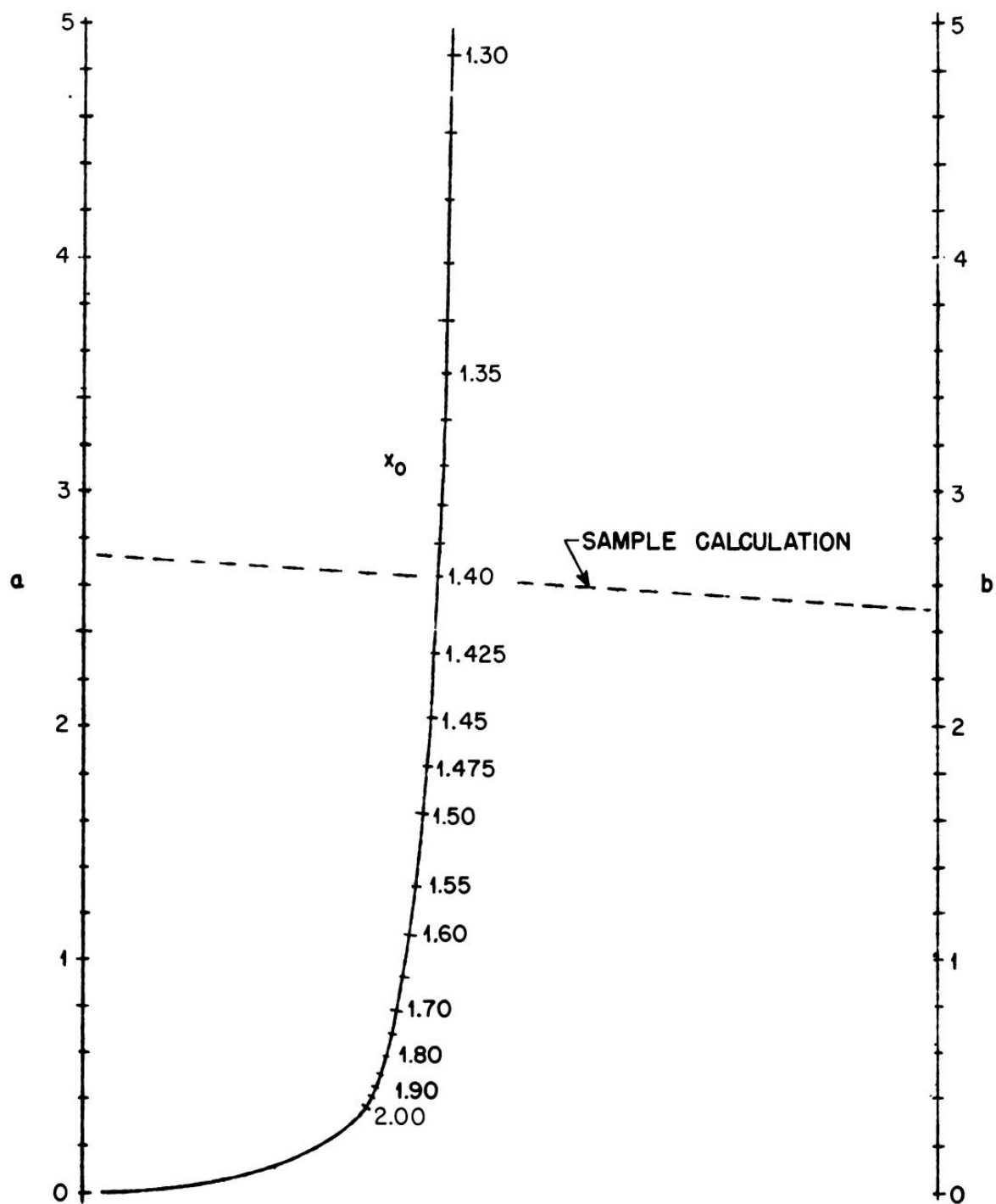


Fig. 1.33—Nomograph for determining x_0 , the optimum ratio of the outside to the inside radius of the coil.

$$a = \frac{C'_{cu}(tr_0f)^2(4Es)^2}{C'_p(10H_0)^2}$$

$$b = \frac{C'_{st}(tr_0f)(4Es)^2R_0^2B}{C'_p(10H_0)^2A}$$

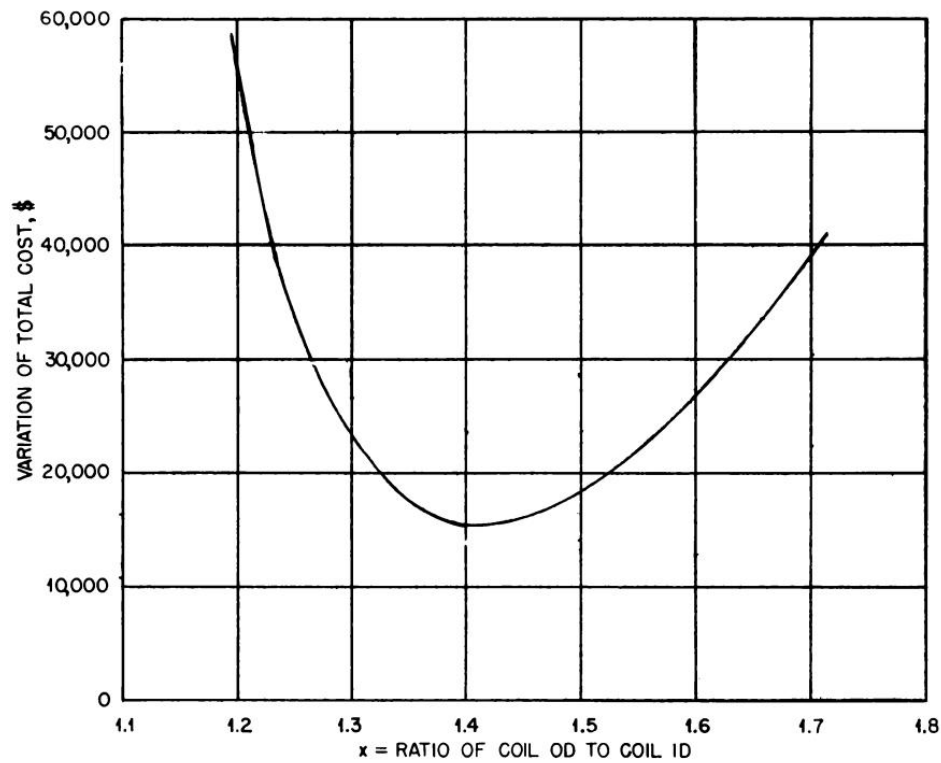


Fig. 1.34—Variation of total cost of a magnet with the outside diameter of the coils. Variation is small over a wide range of diameters near the minimum. Factors not included in the derivation will be important.

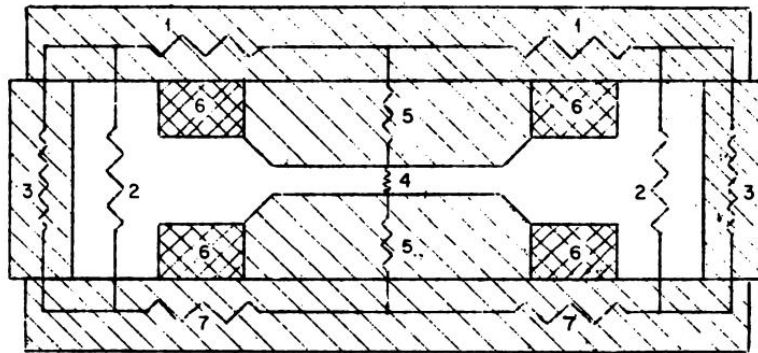


Fig. 1.35—Assumed equivalent circuit for yoke. 1, reluctance of upper yoke; 2, air-leakage-path reluctance; 3, reluctance of vertical yoke; 4, gap reluctance; 5, reluctance of pole; 6, coils supply magnetomotive force; 7, reluctance of lower yoke.

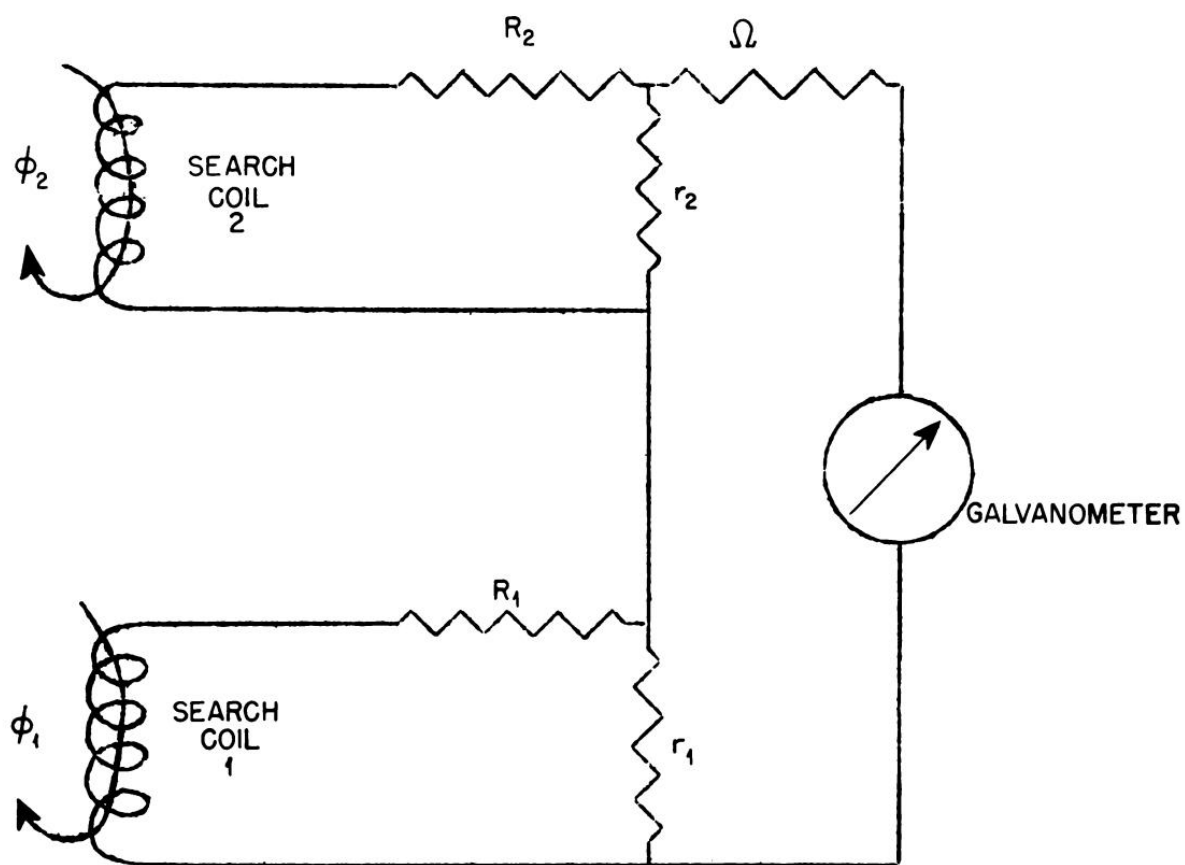


Fig. 1.36—Basic circuit used in comparing two quantities of flux; R_1 and R_2 include search coils and leads.

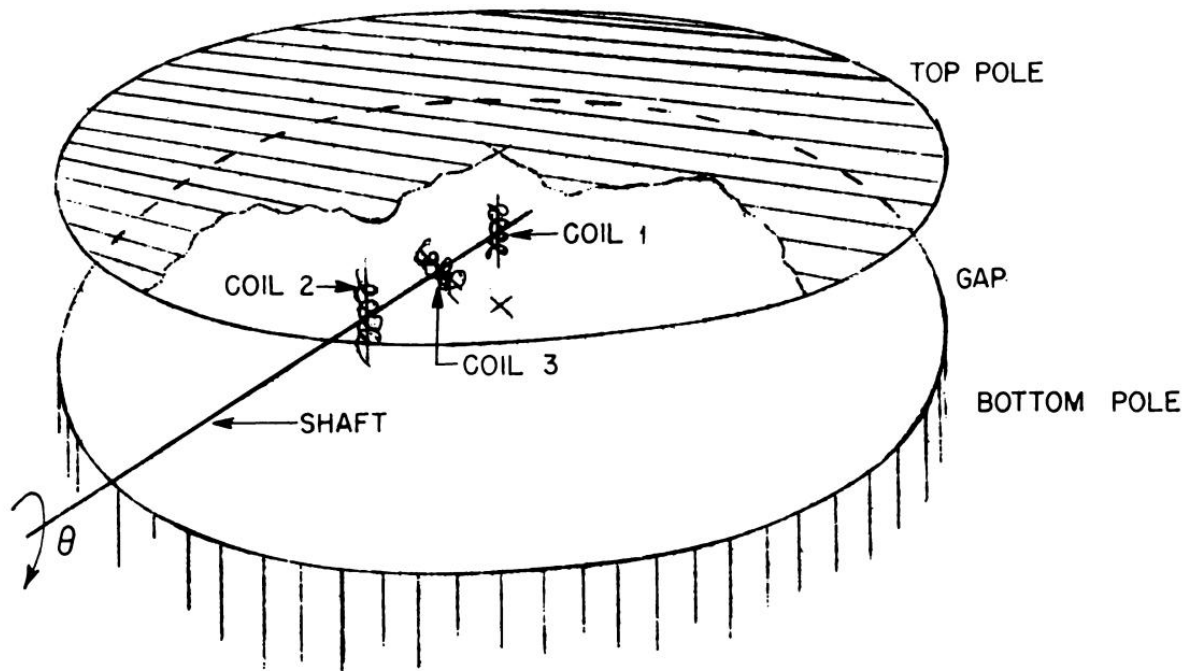


Fig. 1.37a—Field-shape measurement; arrangement of coils on a shaft lying in the mid-plane of the gap.

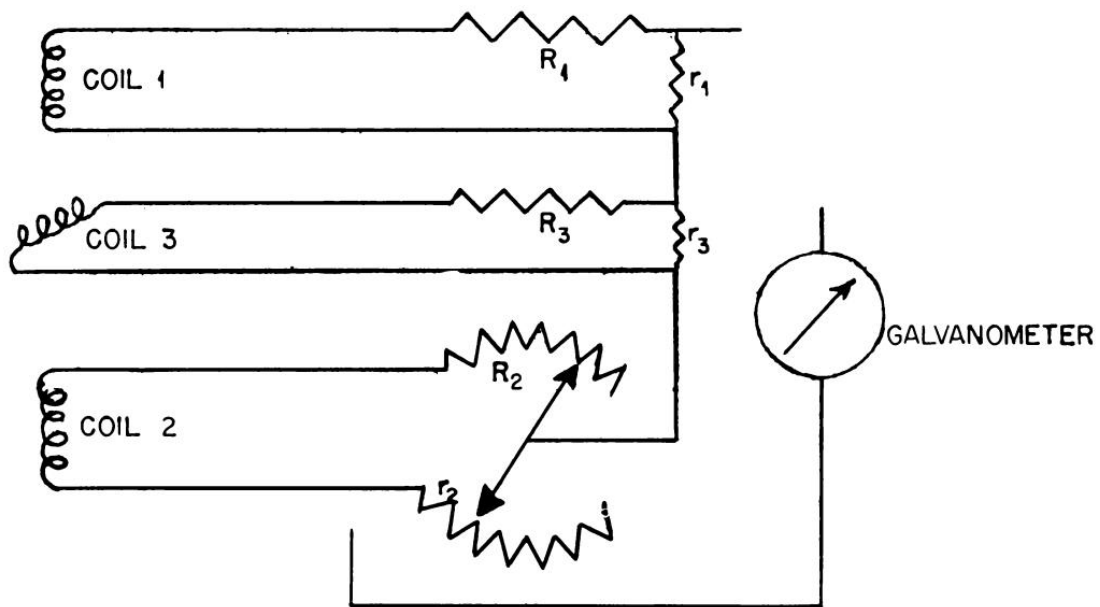


Fig. 1.37b—Field-shape measurement; arrangement of the circuit.

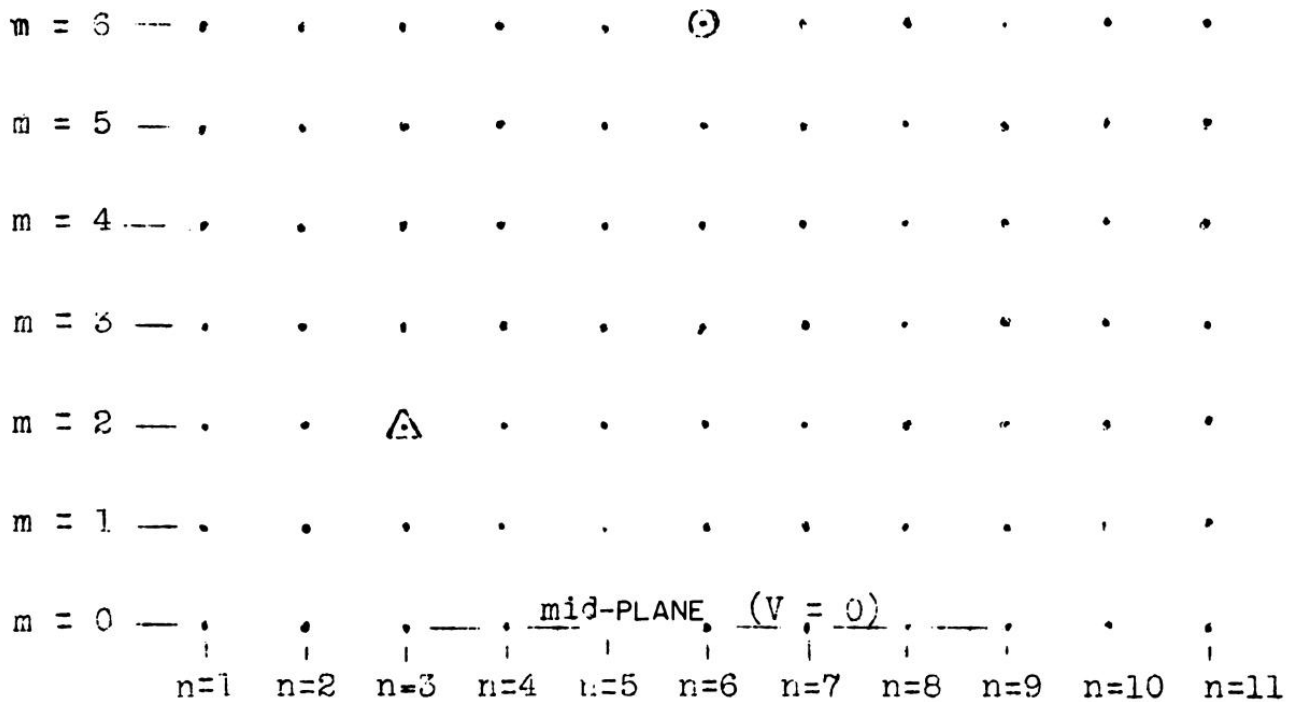


Fig. 1.38—Numerical calculation of potential; the potential at points off the mid-plane may be calculated numerically from the values of the field in the mid-plane, measured at a set of equidistant points on a line. For example, at the point $m = 6, n = 6$, the potential is given by

$$V_{6,6} = 2084(H_6) - 1569(H_5 + H_7) + 672(H_4 + H_8) - 161(H_3 + H_9) + 20(H_2 + H_{10}) - (H_1 + H_{11})$$

and at the point $m = 2, n = 3$,

$$V_{2,3} = 4(H_3) - (H_2 + H_4)$$

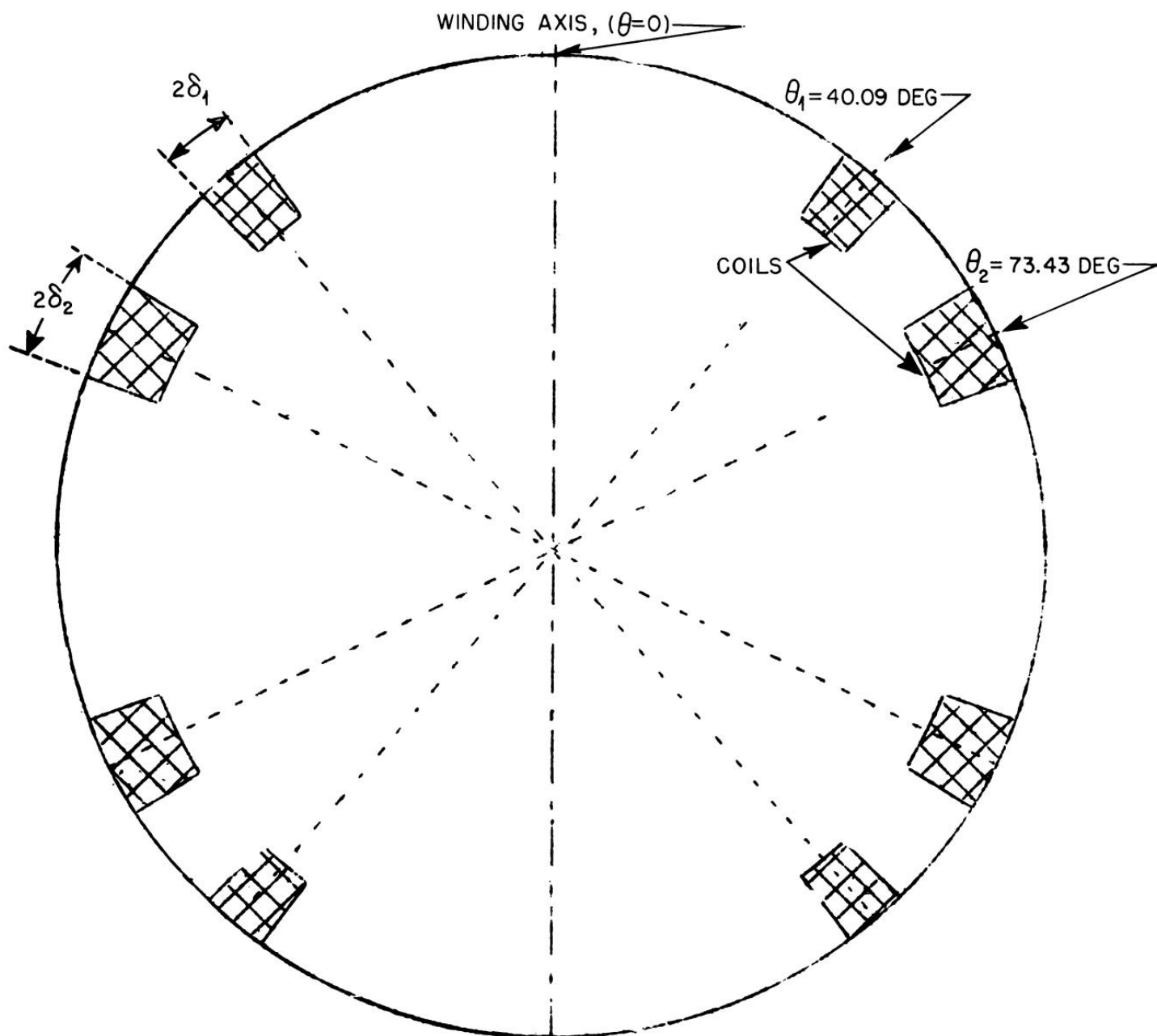


Fig. 1.41—Cross section of search coil that accurately measures the field at its center.

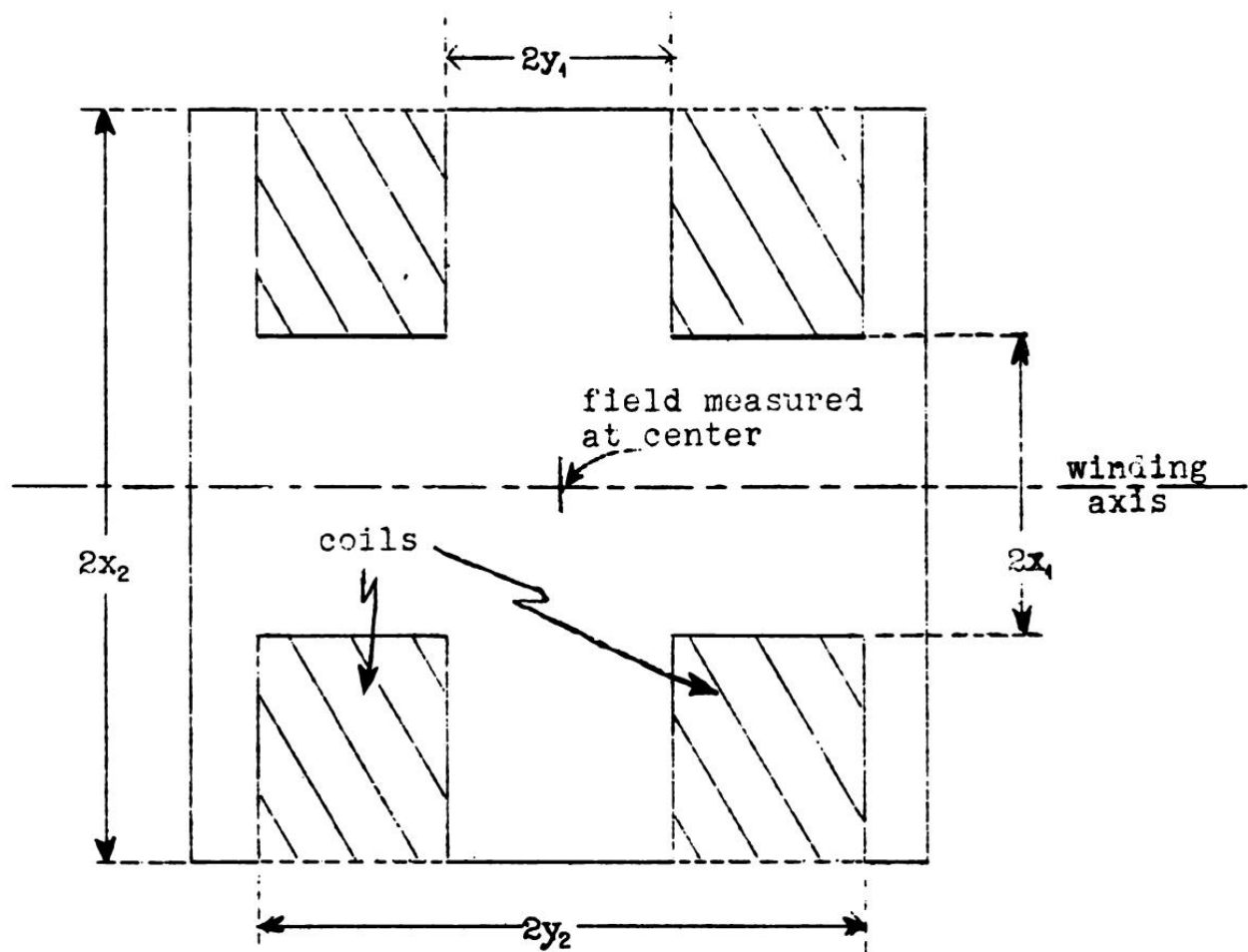


Fig. 1.42—Search coil, convenient to wind, measures the component of the field at its center parallel to the winding axis; shown in cross section.

Technical Report No. 2

DESIGN OF CLOUD-CHAMBER COILS

By M. H. Foss

The field coils for a cloud chamber are required to have no windings over the mid-plane in which the chamber is located and are required to produce a uniform field. In this paper the problem of producing a uniform field by Helmholtz-type coils is considered. Helmholtz coils have cross sections small in comparison with their radii squared, and thus require excessive power where a high field is required. The coils considered below have cross sections comparable with their radii squared and have reasonable power requirements.

It is clear that any uniformity condition can be met by making the coils sufficiently large. However, this approach results in bulky inefficient coils. The problem of the uniformity of the field can be studied by making a power-series expansion in terms of the radius, u , of the field point in the mid-plane. This power series can be developed from the expression for the field of a loop of wire,

$$dH_{\theta} = \frac{2\pi j \sin \alpha}{a} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{u}{a}\right)^{n-1} P_n^1(\cos \alpha) P_n^1(\cos \theta) ds \quad (1)$$

where dH_{θ} = part of field developed by element of the coil, ds

ds = area of the wire

j = current density in the wire

a = radial coordinate of the wire (spherical, polar)

α = colatitude of the wire

u = radial coordinate of the field point

θ = colatitude of the field point

For the field in the mid-plane due to a coil symmetric about the mid-plane, this equation becomes

$$dH = \frac{2\pi j \sin \alpha}{a} \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2 (-4)^m} \left(\frac{u}{a}\right)^{2m} P_{(2m+1)}^1(\cos \alpha) \quad (2)$$

The boundaries of coils that are simple to wind are more easily expressed in cylindrical coordinates; therefore, the first four terms of this equation are given in cylindrical coordinates:

$$\left. \begin{aligned} dH_0 &= 2\pi j \frac{r^2}{(r^2 + Z^2)^{3/2}} ds \\ \frac{dH_1}{U^2} &= -\frac{3}{2} \pi j \frac{r^2}{(r^2 + Z^2)^{7/2}} (4Z^2 - r^2) ds \\ \frac{dH_2}{U^4} &= \frac{45}{32} \pi j \frac{r^2}{(r^2 + Z^2)^{11/2}} (8Z^4 - 12Z^2 r^2 + r^4) ds \\ \frac{dH_3}{U^6} &= -\frac{35}{128} \pi j \frac{r^2}{(r^2 + Z^2)^{15/2}} (64Z^6 - 240r^2 Z^4 + 120Z^2 r^4 - 5r^6) ds \end{aligned} \right\} \quad (3)$$

where z and r are the cylindrical variables. The higher terms can be similarly expressed but they are less useful. Either Eq. 2 or Eqs. 3 can be used for a coil of any shape and with any current distribution.

Equations 3, when integrated with respect to z , assuming that j is not a function of z , are

$$\left. \begin{aligned} dH_0 &= 2\pi jZ \frac{dr}{(r^2 + Z^2)^{1/2}} \Big|_{Z \text{ limits}} \\ \frac{dH_1}{U^2} &= -\frac{3}{2} \pi jZ \frac{r^2 dr}{(r^2 + Z^2)^{5/2}} \Big|_{Z \text{ limits}} \\ \frac{dH_2}{U^4} &= \frac{15}{32} \pi jZ \frac{r^2(3r^2 - 4Z^2)}{(r^2 + Z^2)^{9/2}} dr \Big|_{Z \text{ limits}} \\ \frac{dH_3}{U^6} &= -\frac{35}{128} \pi jZ \frac{r^2(-5r^4 + 20r^2Z^2 - 8Z^4)}{(r^2 + Z^2)^{13/2}} dr \Big|_{Z \text{ limits}} \end{aligned} \right\} \quad (4)$$

Assuming that j is not a function of r , Eqs. 4 may be integrated with respect to r .

$$\left. \begin{aligned} H_0 &= \left[\begin{aligned} &2\pi jZ \log \left(\frac{r}{Z} + \sqrt{1 + \frac{r^2}{Z^2}} \right) \Big|_Z \\ &2\pi jZ \sinh^{-1} \frac{r}{Z} \Big|_Z \end{aligned} \right] r \\ \frac{H_1}{U^2} &= -\frac{1}{2} \pi j \frac{r^3}{Z(r^2 + Z^2)^{3/2}} \Big|_Z \Big|_r \\ \frac{H_2}{U^4} &= -\frac{1}{32} \pi j \frac{r^3(20Z^4 + 7Z^2r^2 + 2r^4)}{Z^3(Z^2 + r^2)^{7/2}} \Big|_Z \Big|_r \\ \frac{H_3}{U^6} &= \frac{1}{384} \pi j \frac{r^3(280Z^8 + 28Z^6r^2 + 99r^4Z^4 + 44Z^2r^6 + 8r^8)}{Z^5(Z^2 + r^2)^{11/2}} \Big|_Z \Big|_r \end{aligned} \right\} \quad (5)$$

Equations 5 describe the field produced by a restricted class of coils, which are illustrated in Fig. 2.1. More complicated shapes or current distributions are justified only if a very high field is required.

The labor associated with a complete study of Eqs. 5 is excessive and will not be attempted here. It is important to note that

$$\rho^{-2n} \frac{H_n}{H_0} < \frac{3}{4} \quad n = 1, 2, 3 \dots \quad (6)$$

where ρ is the ratio of u to r_0 , the inside radius of the coil. Thus it is not possible for any term of the power series for H to have an extremely large coefficient. Furthermore, the coefficients of higher terms tend to become smaller as is shown in Fig. 2.2. It is therefore reasonable to make H_1 identically zero without studying the remaining terms.

The equation for H_1 may be written

$$\frac{1}{\rho^2} H_1 = -\pi j r_0 \left(\frac{y^3}{v(y^2 + v^2)^{3/2}} - \frac{1}{v(1 + v^2)^{3/2}} - \frac{y^3}{w(y^2 + w^2)^{3/2}} + \frac{1}{w(1 + w^2)^{3/2}} \right) \quad (7)$$

where $y = r_1/r_0$

$v = z_2/r_0$

$w = z_1/r_0$

r_1 = outside radius of the coil

z_2 = one-half the axial length of the coil

z_1 = one-half the gap at the center of the coil

(see Fig. 2.1). H_1 will vanish if the parentheses in Eq. 7 vanish; this condition may be written

$$0 = \frac{\frac{1}{y}}{\left(1 + \frac{v^2}{y^2}\right)^{3/2}} - \frac{1}{v(1 + v^2)^{3/2}} - \frac{\frac{1}{y}}{\left(1 + \frac{w^2}{y^2}\right)^{3/2}} + \frac{1}{w(1 + w^2)^{3/2}}$$

and if the function $F(x)$ is defined as

$$F(x) = \frac{1}{x(1 + x^2)^{3/2}} \quad (8)$$

then the condition for the vanishing of H_1 becomes

$$0 = \frac{1}{y} F\left(\frac{v}{y}\right) - F(v) - \frac{1}{y} F\left(\frac{w}{y}\right) + F(w) \quad (9)$$

In order to simplify the solution of this equation, Table 2.1 was constructed. This table gives $\log F(x)$ as a function of $\log(x)$. A sample calculation is given in Table 2.2.

Equations 5 are given below in the notation of the previous paragraph. This notation is used because it clearly indicates the role of the scale or size of the coil. The equations for the power required by the coil for the volume of conductor in the coil are also given. Previously, j has stood for the average current density (abamp) in the coil; in the following equations it will stand for the current density in the conductor (amperes) and the "space factor" f will be introduced to give the density of conductor in the coil.

$$H_0 = \frac{4\pi f j r_0}{10} A$$

$$A = V \log_e \left(\frac{Y + \sqrt{V^2 + Y^2}}{1 + \sqrt{V^2 + 1}} \right) - W \log_e \left(\frac{Y + \sqrt{W^2 + Y^2}}{1 + \sqrt{W^2 + 1}} \right)$$

$$\frac{1}{\rho^2} \frac{H_1}{H_0} = -\frac{1}{4A} \left(\frac{Y^3}{V(Y^2 + V^2)^{3/2}} - \frac{1}{V(1 + V^2)^{3/2}} - \frac{Y^3}{W(Y^2 + W^2)^{3/2}} + \frac{1}{W(1 + W^2)^{3/2}} \right)$$

$$\frac{1}{\rho^4} \frac{H_2}{H_0} = -\frac{1}{64A} \left(\frac{Y^3(20V^4 + 7V^2Y^2 + 2Y^4)}{V^3(V^2 + Y^2)^{7/2}} - \frac{20V^4 + 7V^2 + 2}{V^3(V^2 + 1)^{7/2}} - \frac{Y^3(20W^4 + 7WY^2 + 2Y^4)}{W^3(W^2 + Y^2)^{7/2}} + \frac{20W^4 + 7W^2 + 2}{W^3(W^2 + 1)^{7/2}} \right)$$

$$\frac{1}{\rho^6} \frac{H_3}{H_0} = \frac{1}{768A} \left(\frac{Y^3 B(V, Y)}{V^5(V^2 + Y^2)^{11/2}} - \frac{B(V, 1)}{V^5(V^2 + 1)^{11/2}} - \frac{Y^3 B(W, Y)}{W^5(W^2 + Y^2)^{11/2}} + \frac{B(W, 1)}{W^5(W^2 + 1)^{11/2}} \right)$$

where $B(a, b) = 280a^8 + 28a^6b^2 + 99a^4b^4 + 44a^2b^6 + 8b^8$

$V = 2\pi f r_0^3 (V - W)(Y^2 - 1)$

$P = j^2 \Omega V$

where V = volume of conductor

P = power dissipated in coil (steady current) (watts)

Ω = resistivity of conductor (ohm cm)

When W is small, the terms expressing the homogeneity of the field are conveniently written as

$$\frac{1}{\rho^2} \frac{H_1}{H_0} = \frac{1}{4A} \frac{Y^3}{V(V^2 + Y^2)^{3/2}} - \frac{1}{V(1 + V^2)^{3/2}} - \frac{3!}{2^2} W(1 - Y^{-2}) + \frac{5!}{2^4(2!)^2} W^3(1 - Y^{-4}) - \frac{7!}{2^6(3!)^2} W^5(1 - Y^{-6}) + \dots$$

$$+ (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2} W^{(2n-1)}(1 - Y^{-2n}) + \dots$$

$$\frac{1}{\rho^4} \frac{H_2}{H_0} = -\frac{1}{64A} \left\{ \frac{Y^3(20V^4 + 7V^2Y^2 + 2Y^4)}{V^3(V^2 + Y^2)^{7/2}} - \frac{20V^4 + 7V^2 + 2}{V^3(V^2 + 1)^{7/2}} + \frac{45}{4} W(1 - Y^{-4}) - \frac{175}{4} W^3(1 - Y^{-6}) \right.$$

$$\left. + \frac{6615}{64} W^5(1 - Y^{-8}) - \dots \right.$$

$$\frac{1}{\rho^6} \frac{H_3}{H_0} = \frac{1}{768A} \frac{Y^3 B(V, Y)}{V^5(V^2 + Y^2)^{11/2}} - \frac{B(V, 1)}{V^5(V^2 + 1)^{11/2}} - \frac{175}{2} W(1 - Y^{-6}) + \frac{11025}{16} W^3(1 - Y^{-8})$$

$$- \frac{43659}{16} (1 - Y^{-10}) + \dots$$

The first of these, $(1/\rho^2)(H_1/H_0)$, may be used when Table 2.1 does not contain the required value of $F(x)$.

Apart from the scale (or size) of the coils, there are three parameters, W , V , and Y , which can be used to adjust the characteristics of the system. The condition for eliminating the dependence of the field on ρ^2 leaves only two parameters free. Something of the wide variety of coils which can be obtained is indicated in Tables 2.3 and 2.4, where two examples are given. It is perhaps worth noting that r_{eff} can be improved by allowing $(1/\rho)(H_1/H_0)$ to have a suitable positive value which will partly compensate for the ρ^4 term. This can be done by varying V , since the higher terms are insensitive to V . In any case a considerable amount of numerical work is required to obtain the best design for particular circumstances.

Log (x)	Log F(x)	Log (x)	Log F(x)	Log (x)	Log F(x)	Log (x)	Log F(x)	Log (x)	Log F(x)
0.70 - 2	1.29836	0.10 - 1	0.88975	0.50 - 1	0.43791	0.90 - 1	0.7813 - 1	0.30	0.6540 - 2
0.71 - 2	1.28827	0.11 - 1	0.87927	0.51 - 1	0.42512	0.91 - 1	0.7596 - 1	0.31	0.6199 - 2
0.72 - 2	1.27818	0.12 - 1	0.86877	0.52 - 1	0.41222	0.92 - 1	0.7375 - 1	0.32	0.5856 - 2
0.73 - 2	1.26809	0.13 - 1	0.85825	0.53 - 1	0.39920	0.93 - 1	0.7150 - 1	0.33	0.5511 - 2
0.74 - 2	1.25800	0.14 - 1	0.84770	0.54 - 1	0.38604	0.94 - 1	0.6923 - 1	0.34	0.5164 - 2
0.75 - 2	1.24791	0.15 - 1	0.83713	0.55 - 1	0.37275	0.95 - 1	0.6692 - 1	0.35	0.4815 - 2
0.76 - 2	1.23782	0.16 - 1	0.82653	0.56 - 1	0.35932	0.96 - 1	0.6457 - 1	0.36	0.4464 - 2
0.77 - 2	1.22773	0.17 - 1	0.81590	0.57 - 1	0.34575	0.97 - 1	0.6219 - 1	0.37	0.4111 - 2
0.78 - 2	1.21763	0.18 - 1	0.80525	0.58 - 1	0.33204	0.98 - 1	0.5978 - 1	0.38	0.3756 - 2
0.79 - 2	1.20752	0.19 - 1	0.79457	0.59 - 1	0.31817	0.99 - 1	0.5733 - 1	0.39	0.3400 - 2
0.80 - 2	1.19741	0.20 - 1	0.78385	0.60 - 1	0.30416	0.00	0.5485 - 1	0.40	0.3041 - 2
0.81 - 2	1.18729	0.21 - 1	0.77310	0.61 - 1	0.28998	0.01	0.5233 - 1	0.41	0.2682 - 2
0.82 - 2	1.77717	0.22 - 1	0.76232	0.62 - 1	0.27563	0.02	0.4978 - 1	0.42	0.2320 - 2
0.83 - 2	1.16703	0.23 - 1	0.75150	0.63 - 1	0.26111	0.03	0.4719 - 1	0.43	0.1958 - 2
0.84 - 2	1.15691	0.24 - 1	0.74064	0.64 - 1	0.24640	0.04	0.4457 - 1	0.44	0.1593 - 2
0.85 - 2	1.14675	0.25 - 1	0.72974	0.65 - 1	0.23150	0.05	0.4191 - 1	0.45	0.1227 - 2
0.86 - 2	1.13659	0.26 - 1	0.71879	0.66 - 1	0.21640	0.06	0.3923 - 1	0.46	0.0860 - 2
0.87 - 2	1.12643	0.27 - 1	0.70781	0.67 - 1	0.20111	0.07	0.3651 - 1	0.47	0.0492 - 2
0.88 - 2	1.11626	0.28 - 1	0.69678	0.68 - 1	0.18562	0.08	0.3375 - 1	0.48	0.0122 - 2
0.89 - 2	1.10609	0.29 - 1	0.68570	0.69 - 1	0.16992	0.09	0.3096 - 1	0.49	0.9751 - 3
0.90 - 2	1.09591	0.30 - 1	0.67458	0.70 - 1	0.15401	0.10	0.2814 - 1	0.50	0.9379 - 3
0.91 - 2	1.08573	0.31 - 1	0.66341	0.71 - 1	0.13788	0.11	0.2528 - 1	0.51	0.9006 - 3
0.92 - 2	1.07553	0.32 - 1	0.65218	0.72 - 1	0.12152	0.12	0.2239 - 1	0.52	0.8631 - 3
0.93 - 2	1.06532	0.33 - 1	0.64090	0.73 - 1	0.10493	0.13	0.1947 - 1	0.53	0.8256 - 3
0.94 - 2	1.05510	0.34 - 1	0.62956	0.74 - 1	0.08809	0.14	0.1652 - 1	0.54	0.7880 - 3
0.95 - 2	1.04487	0.35 - 1	0.61816	0.75 - 1	0.07101	0.15	0.1353 - 1	0.55	0.7502 - 3
0.96 - 2	1.03462	0.36 - 1	0.60669	0.76 - 1	0.05368	0.16	0.1052 - 1	0.56	0.7124 - 3
0.97 - 2	1.02436	0.37 - 1	0.59515	0.77 - 1	0.03609	0.17	0.0748 - 1	0.57	0.6744 - 3
0.98 - 2	1.01410	0.38 - 1	0.58356	0.78 - 1	0.01823	0.18	0.0440 - 1	0.58	0.6364 - 3
0.99 - 2	1.00381	0.39 - 1	0.57189	0.79 - 1	0.00010	0.19	0.0130 - 1	0.59	0.5983 - 3
0.00 - 1	0.99352	0.40 - 1	0.56015	0.80 - 1	0.9817 - 1	0.20	0.9817 - 2	0.60	0.5601 - 3
0.01 - 1	0.98321	0.41 - 1	0.54834	0.81 - 1	0.9630 - 1	0.21	0.9501 - 2	0.61	0.5218 - 3
0.02 - 1	0.97291	0.42 - 1	0.53644	0.82 - 1	0.9440 - 1	0.22	0.9182 - 2	0.62	0.4385 - 3
0.03 - 1	0.96256	0.43 - 1	0.52446	0.83 - 1	0.9248 - 1	0.23	0.8861 - 2	0.63	0.4067 - 3
0.04 - 1	0.95220	0.44 - 1	0.51239	0.84 - 1	0.9052 - 1	0.24	0.8537 - 2	0.64	
0.05 - 1	0.94184	0.45 - 1	0.50023	0.85 - 1	0.8853 - 1	0.25	0.8210 - 2	0.65	0.3681 - 3
0.06 - 1	0.93146	0.46 - 1	0.48797	0.86 - 1	0.8652 - 1	0.26	0.7881 - 2	0.66	0.3295 - 3
0.07 - 1	0.92106	0.47 - 1	0.47561	0.87 - 1	0.8447 - 1	0.27	0.7549 - 2	0.67	0.2909 - 3
0.08 - 1	0.91064	0.48 - 1	0.46315	0.88 - 1	0.8239 - 1	0.28	0.7215 - 2	0.68	0.2522 - 3
0.09 - 1	0.90021	0.49 - 1	0.45058	0.89 - 1	0.8028 - 1	0.29	0.6879 - 2	0.69	0.2134 - 3
								0.70	0.1746 - 3

Table 2.2—Sample Calculation of the Geometry of a Coil Which Produces No H_1 *†

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
W	V	Y	$\frac{1}{Y} F\left(\frac{V}{Y}\right)$	F(V)	$\frac{1}{Y} F\left(\frac{W}{Y}\right)$	F(W)	Col. 4— Col. 5	Col. 6— Col. 7
0.2	2	3	0.2880	0.0447	4.9670	4.7144	0.2433	0.2526
	1.965	3	0.2980	0.0476			0.2504	
	1.950	3	0.3023	0.0487			0.2536	
	1.958	3	0.2999	0.0481			0.2518	
	1.954	3	0.3010	0.0484			0.2526	
0.2010		3			4.9418	4.6882		0.2536

*It is assumed that $W = 0.2$ and $Y = 3.0$.

†The last figure is calculated to determine the tolerance on the most critical dimension, W . The result is $W = 0.200$, $Y = 3.000$, and $V = 1.954$. If $W = 0.2010$, then

$$\frac{1}{\rho^2} \frac{H_1}{H_0} = +0.0003$$

Table 2.3—Two Coil Designs

	Coil 1	Coil 2
W	0.200	0.050
Y	3.000	2.000
V	1.954	2.809
A	0.895	0.838
$\frac{H}{fjr_0}$	1.124	1.053
$\frac{1}{\rho^4} \frac{H_2}{H_0}$	-0.08	-0.011
$\frac{1}{\rho^6} \frac{H_3}{H_0}$	-0.02	-0.0065
$r_{eff} = \begin{cases} \text{radius where} \\ H = 0.99H_0 \\ r_{eff}/r_0 \end{cases}$	0.59	0.98
$\frac{v}{fr_0^3} = \frac{P}{fj^2 Q^3}$	88.2	52.2
$\frac{Pf}{H^2 Q r_0}$	69.8	47.1
$\frac{H}{jf_{coeff}}$	2.06	1.18
$\frac{Pf}{H^2 Q r_{eff}}$	118	53

Table 2.4 — Specific Examples of the Coils in Table 2.2*

		Coil 1	Coil 2
Effective radius of field, cm	r_{eff}	15	15
Field strength, gauss	H	10,000	10,000
Space factor	f	0.7	0.7
Resistivity of conductor, $\times 10^{-6}$ ohm cm	r_0	2	2
Inside radius of coil, cm	r_0	25.4	16.9
Space between coils, cm	$2Wr_0$	10.2	1.6
Outside diameter of coils, cm	$2Yr_0$	153	63
Power required, watts	P	500,000	230,000
Current density, amp/cm ²	j	460	810
Volume of conductor, cm ³	V	1,000,000	170,000
Weight of copper, tons		10	1.7

*These coils represent random examples and are not the result of a systematic search for any particular optimum design.

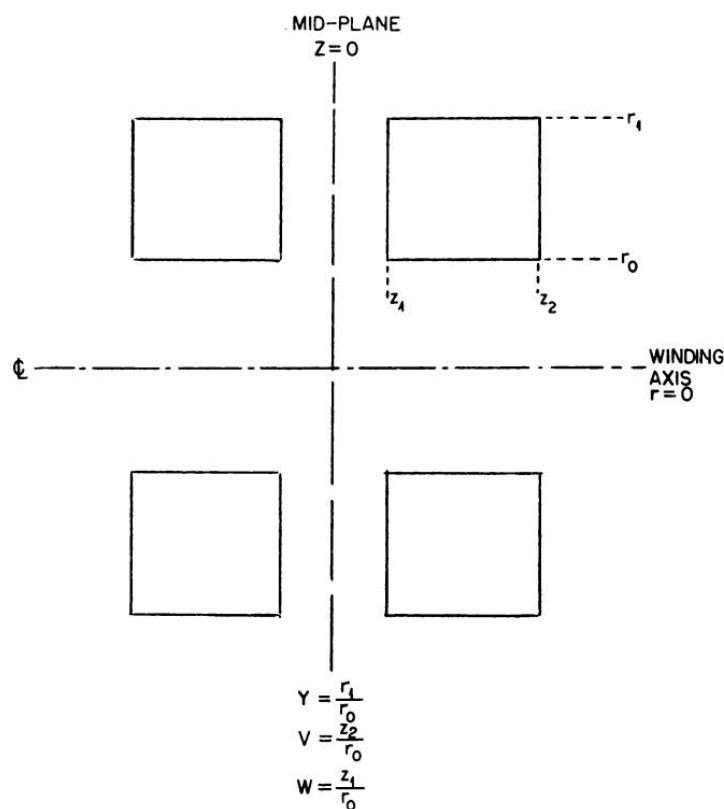


Fig. 2.1 — Coils described by Eqs. 5.

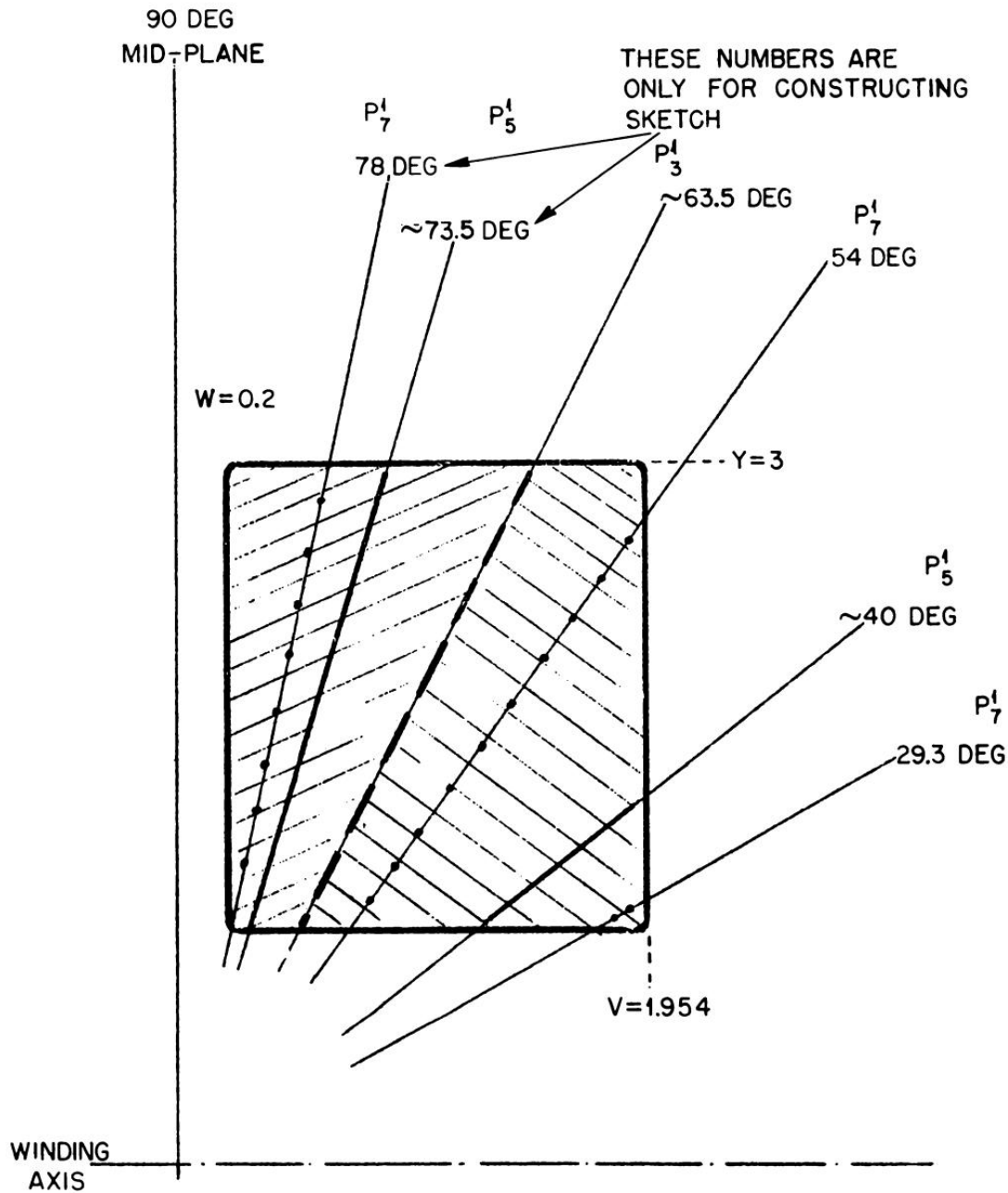


Fig. 2.2—Coil described in Table 2.2 and the nodes of the functions P_3^1 , P_5^1 , and P_7^1 . When a large coil is used, the coefficients associated with the functions P_{2n+1}^1 will tend to vanish, since successive internodal spaces make contributions of opposite sign to the coefficients.

Technical Report No. 3

HIGH MAGNETIC FIELDS

By M. H. Foss

ABSTRACT

It is physically possible to obtain very high magnetic fields, several million gauss. The limitation is the large power requirement.

A good background for the design of practical magnets is found in the study of an "ideal" magnet which has the optimum current-density distribution.

Two designs of coils are discussed. The first comes close to the optimum, and the second makes the region where the field is developed very accessible.

3.1 POWER

The quantity

$$\frac{h}{fj_0r_0} \quad (1)$$

where h = field strength at the center of the magnet, gauss

f = space factor of the conductor

j_0 = maximum current density, amp/cm²

r_0 = inside radius of coil, cm

is a number which is determined by the design of the coil. It does not change if every dimension of a design is changed by the same factor. Thus the field obtained is proportional to the inside radius of the coil and a high field can be obtained by increasing the scale, r_0 , of any design. However,

$$\frac{v}{fr_0^3} \quad \frac{p}{fr_0^3j_0^2\rho} \quad (2)$$

where ρ = resistivity of conductor, are also characteristics of the coil design so that the power p and the volume of conductor v increase with the cube of the inside radius.

It is the very great power requirement which necessitates careful study of the design of high-field magnets. The first consideration is therefore that of obtaining the required field with a minimum power. Since this may involve an infinite amount of conductor it is necessary to make this condition read: The minimum power with a given amount of conductor.

It is easily shown that the current-density distribution which produces a given field with the least amount of power is

$$j_\phi = \frac{k \sin \theta}{r^2} \quad j_\theta = 0 \quad j_r = 0 \quad (3)$$

where r, θ, ϕ = spherical polar coordinates

k = constant

j = current density amp/cm²

(The field is produced at $r = 0$ in the direction of the polar axis.) The shape of the outside of the coil which requires the least conductor for a given field and power is given by

$$r^2 = k' \sin \theta \quad (4)$$

The space required for the apparatus which is to be exposed to the field must be free of a conductor. This space should be kept at a minimum. A coil of this type with a spherical cavity at the center is illustrated in Fig. 3.1.

The quantities of interest connected with this ideal coil are

$$\begin{aligned} \frac{h}{fj_0r_0} &= 0.838 \left(1 - \frac{1.073}{S}\right) \\ \frac{p}{fr_0^3j_0^2\rho} &= 8.38 \left(1 - \frac{1.073}{S}\right) \\ \frac{pf}{h^2pr_0} &= \frac{11.93}{1 - \frac{1.073}{S}} \\ \frac{v}{fr_0^3} &= 3.00S^3 - 4.19 \end{aligned} \quad (5)$$

where S = outside radius divided by inside radius. (These results are approximate and are correct to the indicated accuracy when $S > 2.5$. In this and the following work the space factor, i.e., volume of conductor divided by volume of coil, and the resistivity are assumed to be constant over the coil.)

3.2 CURRENT DENSITY

Another number which depends on the design is

$$\frac{pf^2j_0}{h^3} \quad (6)$$

For the coil design discussed above this is

$$\frac{pj_0}{h^3} \frac{f^2}{\rho} = \frac{14.23}{\left(1 - \frac{1.073}{S}\right)^2} \quad (7)$$

The importance of this number may be seen by considering the example, $S = 10$; therefore, $(pf^2j_0)/(\rho h^3) = 17.9$. If $\rho = 2 \times 10^{-6}$ ohm cm, $f = 0.90$, $p = 2 \times 10^6$ watts, and a field of 200,000 gauss is required, then the maximum current density is 1.76×10^5 amp/cm². This is so high that it is impossible to cool the coil if it is to give a steady field (at least with a space factor of 0.9). It is thus necessary to consider the current-density number, $(pj_0f^2)/(h^3\rho)$ if the magnet is to operate continuously.

The best coil for a minimum $(pj_0f^2)/h^3\rho$ is a kernel bounded by

$$r^2 = r_1^2 \sin \theta \quad (8)$$

carrying a constant current density, j_0 , which is surrounded by a coil in which the current is given by

$$j = \frac{\sin \theta}{r^2} r_1^2 j_0 \quad (9)$$

which is bounded by the surface

$$r^2 = r_2^2 \sin \theta \quad (10)$$

For such a coil (spherical hole in center)

$$\begin{aligned} r_0 \frac{h}{f r_0 j_0} &= 0.9 \left(2r_1 - 1.10r_0 - \frac{r_1^2}{r_2} \right) \\ r_0^3 \frac{p}{f j_0^2 r_0^3} &= 3 \left(4r_1^3 - 1.40r_0^3 - \frac{3r_1^4}{r_2} \right) \\ \frac{p j_0^2}{h^3 \rho} &= 2.06 \frac{\left(r_1^3 - 0.35r_0^3 - 0.75 \frac{r_1^4}{r_2} \right)}{\left(r_1 - 0.55r_0 - 0.5 \frac{r_1^2}{r_2} \right)^3} \end{aligned} \quad (11)$$

The lower limit for $(p j_0^2)/h^3$ is obtained by making $r_1^2 = 2.20r_2 r_0$ and

$$r_0 \ll r_1 \quad (12)$$

In this case

$$\begin{aligned} \frac{p j_0^2}{h^3} &\rightarrow 2.06 \\ r_1 &\rightarrow 3 \sqrt{\frac{1}{12} \frac{p}{\rho f j_0^2}} \end{aligned} \quad (13)$$

(See Fig. 3.2, the top three curves.) When the maximum current density and the desired field are known, the power p can be determined. This is the smallest power which can give the required field with the given maximum current density. From p and j_0 the outer radius of the kernel, r_1 , can be determined. For example, let $j_0 = 2 \times 10^4$ amp/cm², $\rho = 2 \times 10^{-6}$ ohm/cm, $f = 0.9$, $h = 2 \times 10^5$ gauss; then by Eq. 13, $p = 2 \times 10^6$ watts and $r_1 = 6.1$ cm.

3.3 INSIDE RADIUS

It may be that the inside radius of the coil must be sufficiently large for a particular experiment, so that the maximum current density is no problem. In this case a coil designed according to the principles given in Sec. 3.1 is appropriate. Or it may be that the inside radius can be so small that the maximum current density is the only problem. In this case the principles of Sec. 3.2 are appropriate. Where the required inside radius falls between these two limits, another approach is required.

In this case both j_0 and r_0 are known and the number $h/(f j_0 r_0)$ can be determined. There is a simple infinity of coils with a kernel $j = j_0$ and an outer zone $j = j_0 r_1 (\sin \theta)/r^2$ and an outside radius between

$$r_2 = 1.11 \frac{h}{fj_0} + 1.10r_0 \quad (14)$$

and $r_2 = \text{infinity}$, which give the required $h/(fj_0)$. The power required is a monotone decreasing function of r_2 ; thus, the choice of r_2 gives a proper balance between the power required and the volume of conductor required. This is illustrated in Fig. 3.2.

3.4 ACCESSIBILITY

The previous discussion was confined to magnets with a spherical cavity for apparatus. For some applications greater accessibility of the region where the field is developed is required. This can be obtained by using more power. Several examples are given in Table 3.1.

Table 3.1*

	Spherical cavity at center (for comparison)	$0 \leq \theta \leq \pi/4$ and $3\pi/4 \leq \theta \leq \pi$ free of conductor	$\left(\frac{U}{2} - \delta\right) \leq \theta \leq \left(\frac{U}{2} + \delta\right)$ free of conductor	$\delta = 4.5 \text{ deg}$
$\frac{h}{fr_0j_0}$	$0.838 \left(1 - \frac{1.07}{S}\right)$	$0.740 \left(1 - \frac{1.04}{S}\right)$	$0.838 \left[(1 - 1.5\delta) \frac{1.07}{S} (1 - 1.4\delta)\right]$	$0.740 \left(1 - \frac{1.08}{S}\right)$
$\frac{p}{f\rho j_0^2 r_0^3}$	$8.38 \left(1 - \frac{1.07}{S}\right)$	$7.40 \left(1 - \frac{1.04}{S}\right)$	$8.39 \left[(1 - 1.5\delta) - \frac{1.07}{S} (1 - 1.4\delta)\right]$	$7.40 \left(1 - \frac{1.08}{S}\right)$
$\frac{p^f}{\rho r_0 h^2}$	$11.9 \frac{1}{\left(1 - \frac{1.07}{S}\right)}$	$13.5 \frac{1}{\left(1 - \frac{1.04}{S}\right)}$	$\frac{11.9}{(1 - 1.5\delta) - 1.07 (1 - 1.4\delta)/S}$	$13.5 \frac{1}{\left(1 - \frac{1.08}{S}\right)}$

* θ is angle.

3.5 CONSTRUCTION

The design which has been discussed is impossible to construct owing to the complex current-density distribution. It is fortunate that departures from the ideal design cause only second-order increases in the power required to produce a given field. The design of a practical coil is found by balancing the complexity of construction against the power required to operate the magnet.

A coil made with rectangular zones of constant current density (as illustrated in Fig. 3.3) can come very close to the ideal, if the ratio of the outside radius of the zone to the inside radius is close to one. However, the construction is facilitated by making this ratio much greater than one. Figure 3.3 shows one possible compromise.

The coil consists of a central zone which may be described in cylindrical coordinates by

$$\frac{r_1}{r_0} = 3.095 \quad \frac{Z_1}{r_0} = 1.862 \quad j = j_0 \quad (15)$$

where r_0 = inside radius
 r_1 = outside radius
 Z_1 = half-length of coil

This coil, taken by itself, has the minimum $(pf)/h^2 r_0 \rho$ which can be obtained with a coil of this type,

$$\frac{pf}{h^2 r_0 \rho} = 31.2 \quad \frac{h}{fr_0 j_0} = 1.80 \quad (16)$$

Surrounding this core are zones with

$$\begin{aligned} \text{Outside radius} &= 3.095 \sqrt{3}^n \\ \text{Middle radius} &= 3.095 \sqrt{3}^{n-1} \\ \text{Inside radius} &= \frac{1 + \sqrt{3}}{2} \sqrt{3}^{n-1} \\ \text{Outside Z dimension} &= 1.862 \sqrt{3}^n \\ \text{Inside Z dimension} &= 1.862 \sqrt{3}^{n-1} \\ \text{Current density} &= 0.806 \cdot 3^{-n} j_0 \end{aligned} \quad (17)$$

The current density is so adjusted that $(pf)/(\rho r_0 h^2)$ is a minimum. For this coil,

$$\left. \begin{aligned} \frac{h}{fr_0 j_0} &= 3.24 \left(1 - 0.44 \frac{1}{3^{N/2}} \right) \\ \frac{pf}{\rho r_0^3 h^2 \rho} &= 189 = \left(1 - 0.49 \frac{1}{3^{N/2}} \right) \\ \frac{pf}{\rho r_0 h^2} &= 18.0 \frac{\left(1 - 0.49 \frac{1}{3^{N/2}} \right)}{\left(1 - 0.49 \frac{1}{3^{N/2}} \right)} \end{aligned} \right\} \quad (18)$$

where N = number of zones.

It is interesting to compare this result with the limit as the zones are made thinner and more numerous. In this case,

$$\left. \begin{aligned} \frac{h}{fr_0 j_0} &= 3.25 \left(1 - \frac{1.38}{S} \right) \\ \frac{pf}{\rho r_0 h^2} &= 17.2 \frac{1}{\left(1 - \frac{1.38}{S} \right)} \end{aligned} \right\} \quad (19)$$

The current density in the first zone is $0.5 j_0 = j_1$ and makes $(pf)/(\rho r_0 h^2)$ a minimum.

If a larger $h/(fr_0 j_0)$ is required, the current densities can be readjusted. For example, if $j_n = 3^{-n} j_0$, then

$$\begin{aligned} \frac{h}{fr_0 j_0} &= 3.47 \left(1 - 0.48 \frac{1}{3^{N/2}} \right) \\ \frac{p}{fr_0^3 j_0^2 \rho} &= 220 \left(1 - 0.54 \frac{1}{3^{N/2}} \right) \\ \frac{pf}{h^2 r_0 \rho} &= 18.3 \frac{\left(1 - 0.54 \frac{1}{3^{N/2}} \right)}{\left(1 - 0.48 \frac{1}{3^{N/2}} \right)} \end{aligned} \quad (20)$$

If $N = 4$,

$$\frac{h}{fr_0j_0} = 3.28 \quad \frac{pf}{h^2r_0\rho} = 19.2$$

The merit of these arrangements can be judged by comparing these results with those of Fig. 3.2. Part of the reduction in efficiency can be attributed to the greater accessibility of the region where the field is produced.

The outside coil by itself (i.e., the coil without zone zero) has a cylindrical space in the center with a length $1.20 r_0$. This coil has poorer accessibility, but lower $(pf)/(h^2r_0\rho)$.

$$\frac{h}{fr_0j_0} = 1.62(1 - 3^{-N/2}) \quad \frac{p}{f\rho r_0^3j_0^2} = 36.3 \cdot 1 - 3^{-N/2} \quad \frac{pf}{h^2r_0\rho} = \frac{13.9}{1 - 3^{-N/2}}$$

The variety of designs which can be considered are so numerous that a complete discussion is not possible.

An interesting coil, of which a core of half-angle 45-deg measure from the polar axis must be free of conductor, is obtained by seeking the best coil with the current density a function of the radius only (in cylindrical coordinates). In this case the optimum shape is given by

$$U^2 = \frac{S^2 \sqrt{8Y^4 + S^4} - (2Y^4 + S^4)}{2Y^2} \quad (21)$$

(see Fig. 3.4), and the current-density distribution

$$j = j_0 \frac{1}{Y \sqrt{U^2 + Y^2}} \quad (22)$$

where $U_0 = Z$ dimension of coil (not constant)

Yr_0 = radial variable

Sr_0 = outside radius of coil

r_0 = inside radius of coil

j_0 = maximum current density

In this case

$$\left. \begin{aligned} \frac{h}{fr_0j_0} &= \frac{2\pi \sqrt{2}}{10} \left(1 - \frac{1.15}{S}\right) \\ \frac{p}{f\rho r_0^3j_0^2} &= 4\pi \left(1 - \frac{1.15}{S}\right) \\ \frac{pf}{h^2r_0\rho} &= \frac{100}{2\pi} \frac{1}{\left(1 - \frac{1.15}{S}\right)} \\ \frac{v}{fr_0^3} &= \pi r_0^3 (0.741S^3 - 1.33) \end{aligned} \right\} \quad (23)$$

The shape of this coil can be simplified as is illustrated in Fig. 3.5. The optimum shape in this case is

$$\begin{aligned}
 U &= Y & 1 \leq Y \leq \frac{S}{2.11} \\
 U &= \frac{S}{2.11} & \frac{S}{2.11} \leq Y \leq S
 \end{aligned}
 \tag{24}$$

and the current density

$$j = j_0 \frac{1}{Y \sqrt{U^2 + Y^2}} \tag{25}$$

These give

$$\left. \begin{aligned}
 \frac{h}{fr_0 j_0} &= \frac{2^{3/2} \pi}{10} \left(1 - \frac{1.08}{S} \right) \\
 \frac{pf}{h^2 \rho r_0} &= \frac{100}{2\pi} \frac{1}{\left(1 - \frac{1.08}{S} \right)} \\
 \frac{v}{fr_0^3} &= \pi (0.876 S^3 - 1.33)
 \end{aligned} \right\} \tag{26}$$

This coil can be constructed by winding conductor of width $2U$ about the axis and adjusting the thickness of the conductor to control the current density. This plan of construction is facilitated by keeping the conductor thickness constant in a zone whose outside radius divided by inside radius is R . In this case, the current density is proportional to $1/Y$ within the zone, the value of $h/(fr_0 j_0)$ is increased by $(R \log R)/(R - 1)$, and $(pf)/(h^2 \rho r_0)$ is increased by $(R - 1)^2/(R \log^2 R)$.

3.6 SIZE OF COIL

The determination of the size of the coil may be an economic problem. One example of this will suffice. Consider the coil illustrated in Fig. 3.5. It will be assumed that the costs of the magnet can be divided into three groups. Costs proportional to power can include the cost of power, generator, and cooling. Costs proportional to the volume may include the cost of the conductor, building, and insulation. Some costs do not depend on the size of the coil and do not influence the economic adjustment of the coil size. The equation for the size of the coil is

$$S(S - 1.08) = \frac{4.52}{\pi} \frac{h}{fr_0} \sqrt{\frac{\rho C'_p}{C'_{cu}}} \tag{27}$$

where C'_p = cost proportional to power, per watt

C'_{cu} = costs proportional to volume of conductor, per cm^3

3.7 CONCLUSION

The appropriate design of a magnet depends on the field required. For low fields, $\sim 10^3$ gauss, the $h/(fr_0 j_0)$ required is sufficiently small, so that it is easily obtained, and the power required is sufficiently small, so that a complicated design is seldom required. For fields of 10^5 gauss and above, the situation is quite different; the power and maximum current density become important factors and more complicated designs are useful.

It has been shown that it is possible to closely approach the best that can be done with relatively simple designs. However, the designs discussed are not in a form suitable for construction. Determining the design of a coil which has zones bounded by planes of constants r and Z (cylindrical coordinates) depends on the solution of simultaneous transcendental equations which could be undertaken for each design but not in a general treatment.

The maximum current density which can be used is an important factor in coil design. This quantity must be determined experimentally before any design can be completed.

The use of silver as the conductor in the central region of a coil may be worth while under some circumstances, as also may be cooling with liquid air.

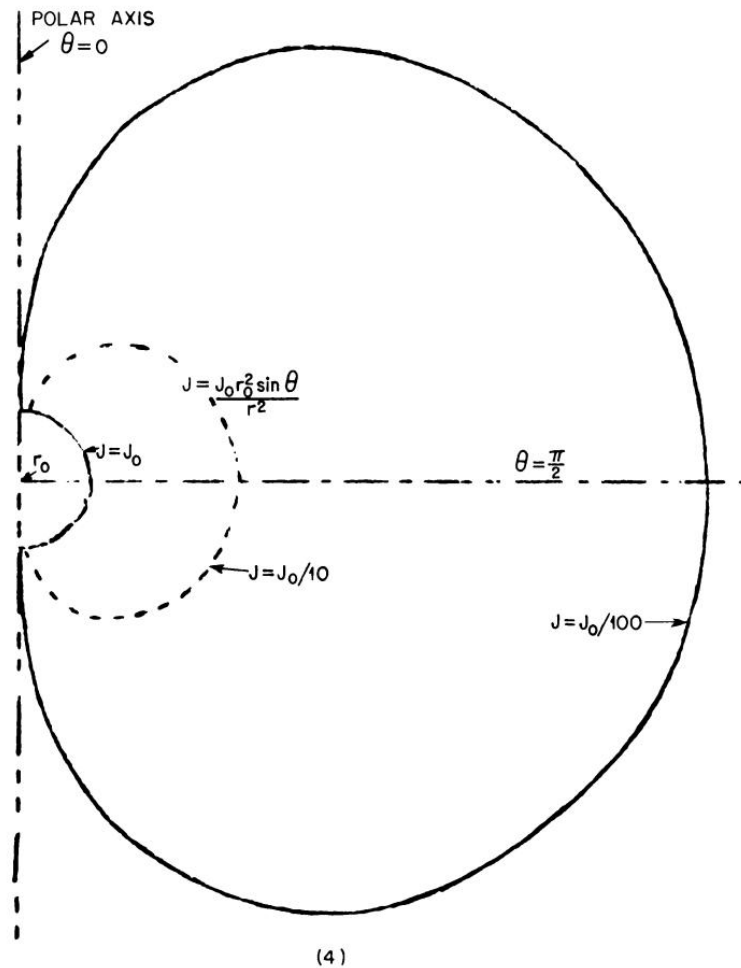


Fig. 3.1—Coil with spherical cavity at center.

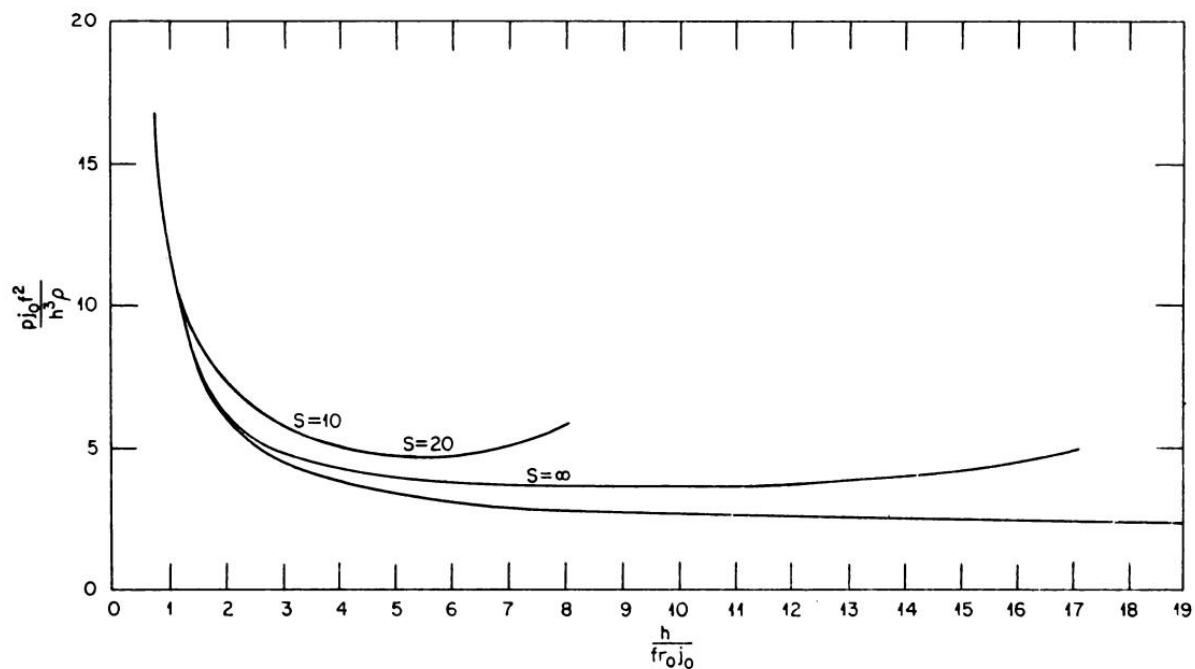


Fig. 3.2a— $(p j_0^2)/h^3$ as a function of $h/(f r_0 j_0)$.

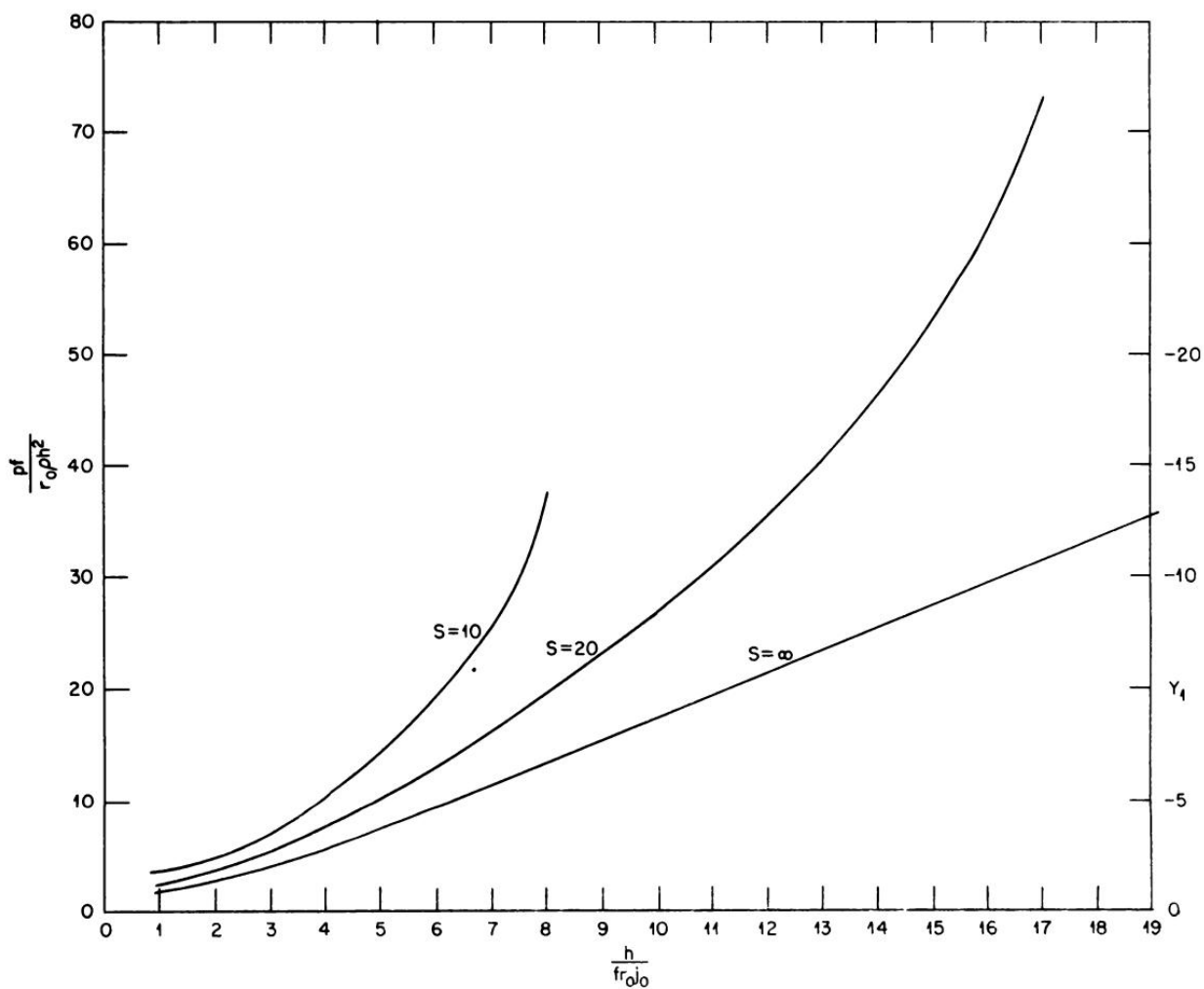


Fig. 3.2b— Y_1 as a function of $h/(f r_0 j_0)$.

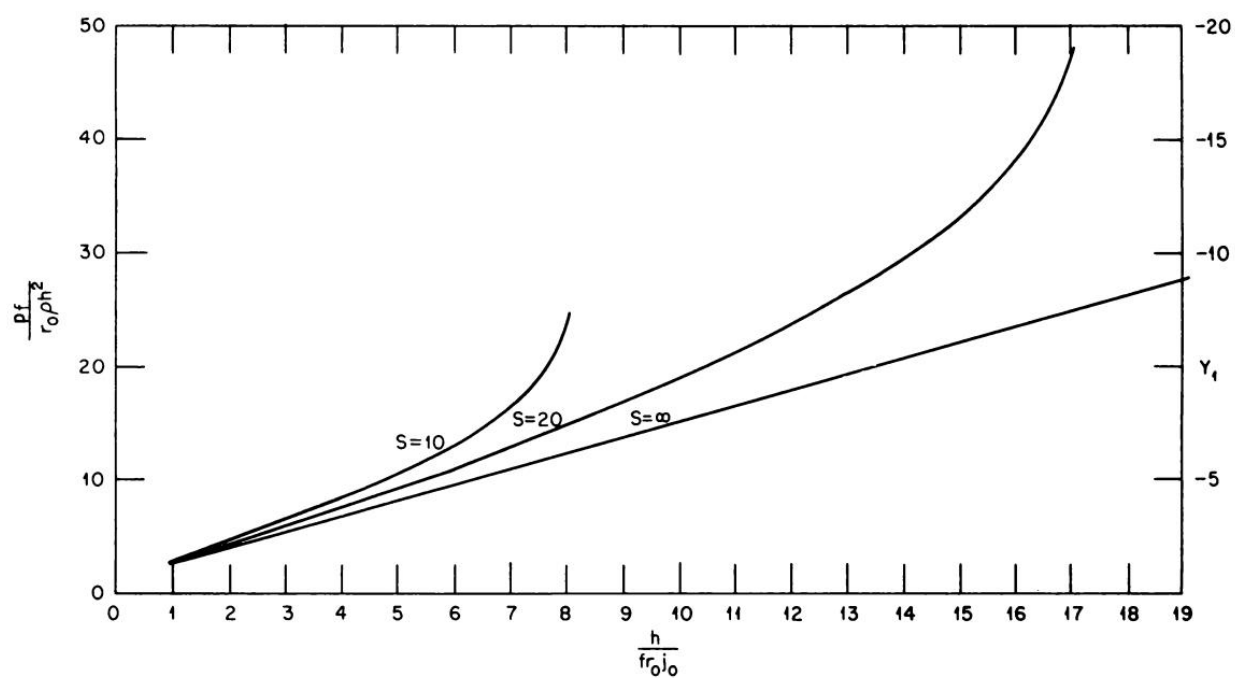


Fig. 3.2c — $(pf)/(h^2 r_0)$ as a function of $h/(f r_0 j_0)$.

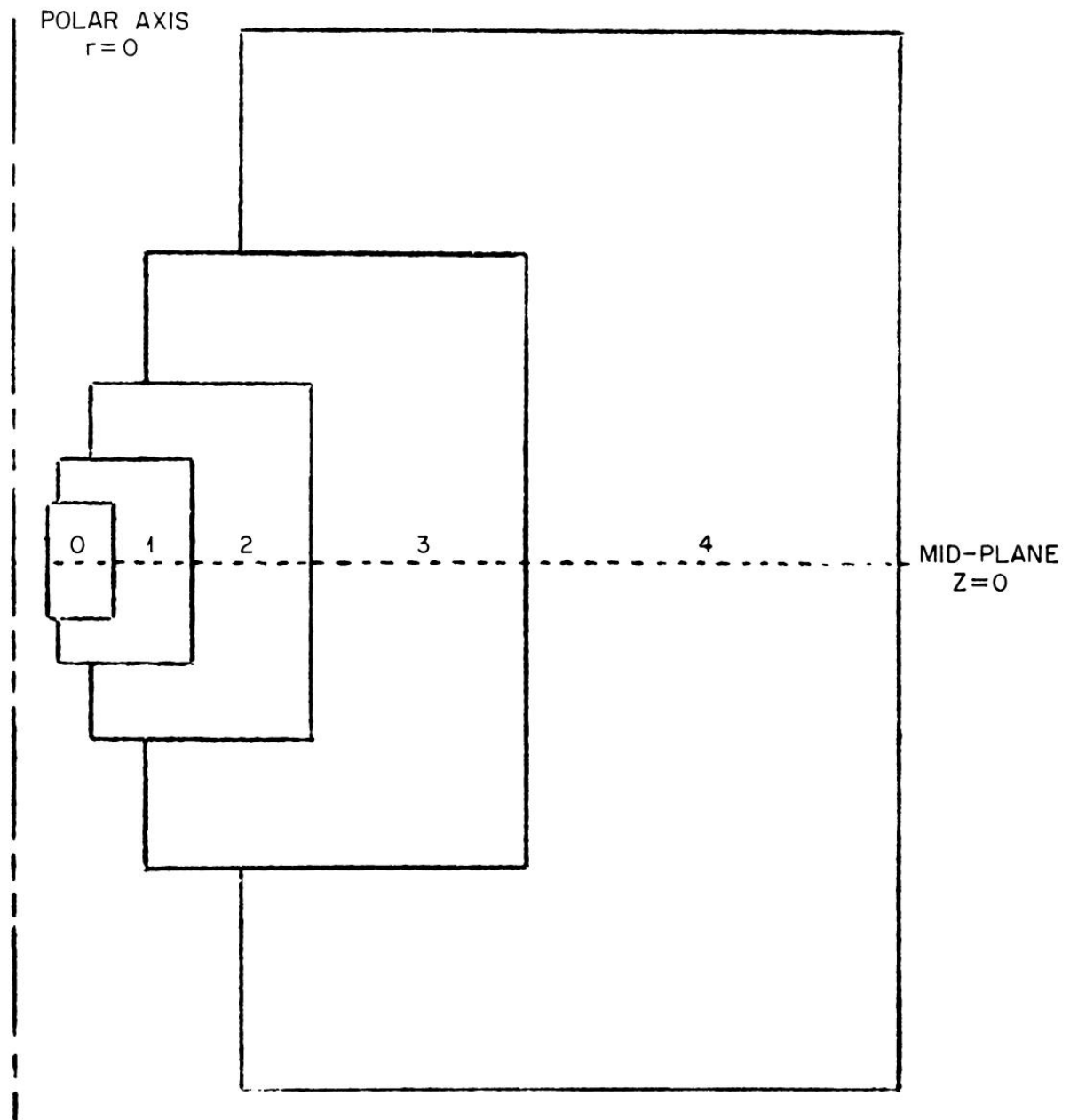


Fig. 3.3—Coil with rectangular zones of constant current density.

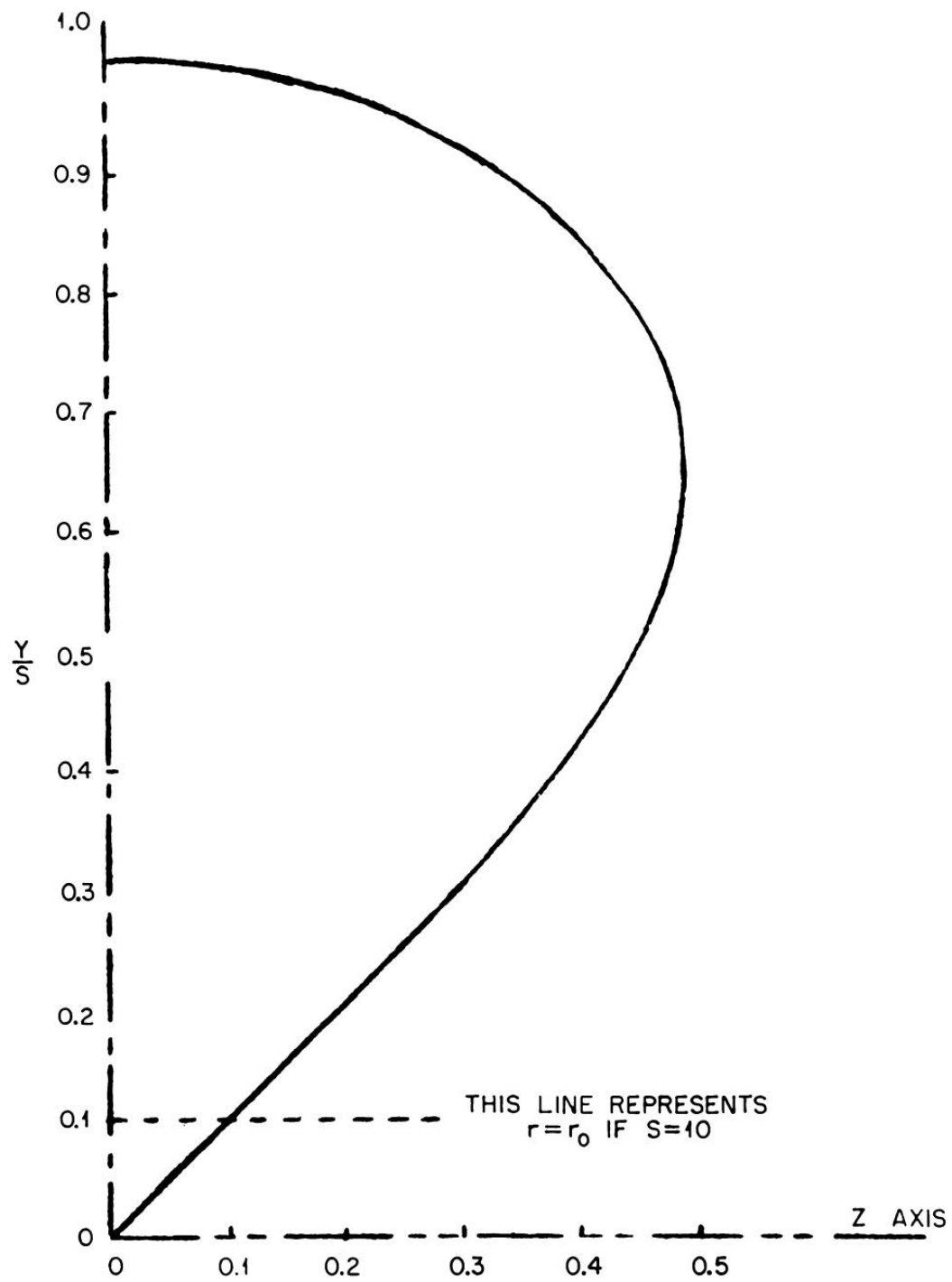


Fig. 3.4—Optimum shape of coil.

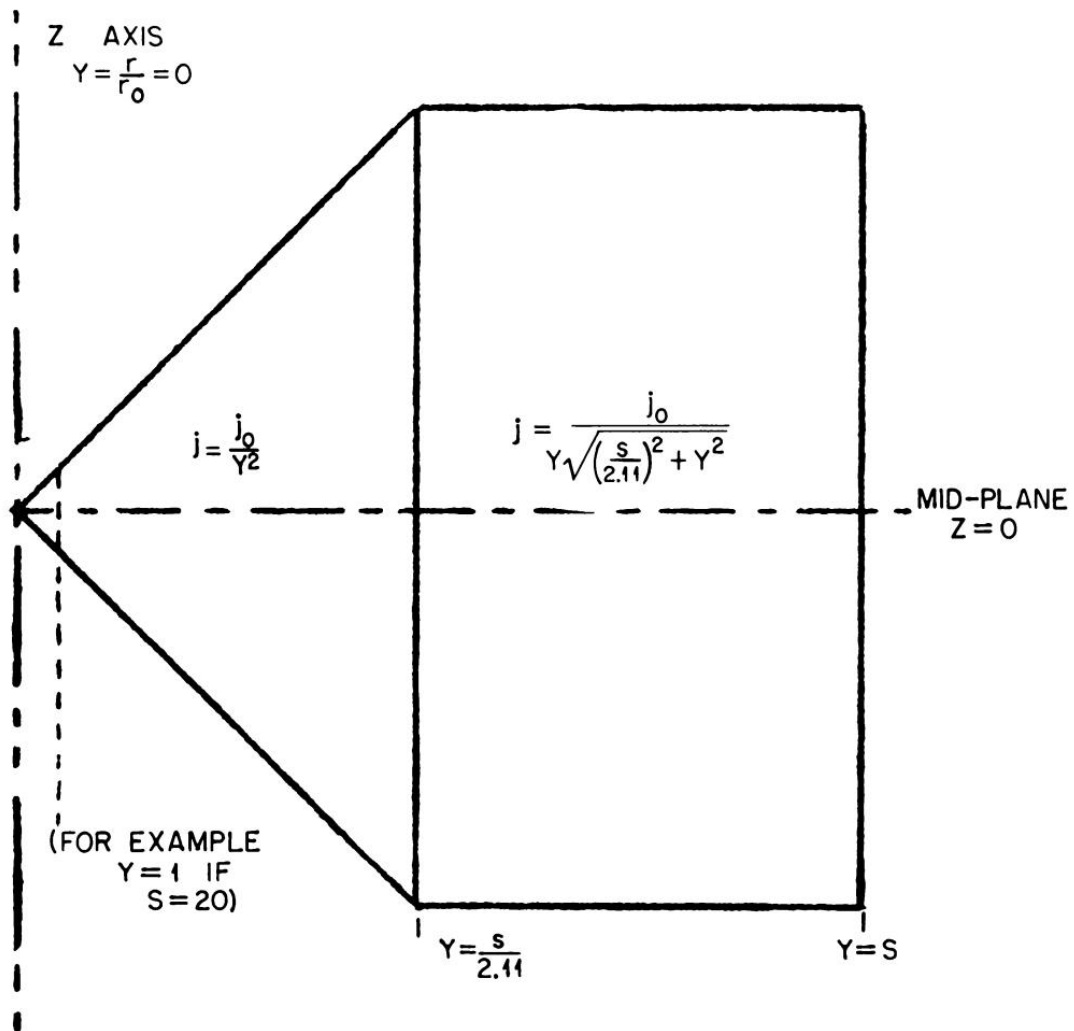


Fig. 3.5—Simplified shape of coil.

DESIGN OF CYCLOTRON VACUUM SYSTEM

By D. Rose

ABSTRACT

The attainment of sufficiently low pressures within the dee and vacuum chamber is, of course, of prime importance in the successful operation of the cyclotron. However, methods for computing the pumping speeds of complex geometries, such as are involved here, leave much to be desired, and in this report no attempt is made to reach an exact solution, but rather to obtain a reasonable approximation of the vacuum equipment required.

Since most of the components of the cyclotron which must be evacuated are of noncircular cross section, a convenient expression for the conductance of such ducts in the range of free molecular flow is derived in Appendix I. As is emphasized there, this expression gives only an approximation to the true conductance, but since the resulting values of conductance are too low (cf. Table 4.2, Appendix I), and in view of the uncertainty in the quantity of gas to be expected from outgassing of exposed surfaces, it is just as well to use these values which will result in pump capacities on the safe side. Therefore, in using the expression for the conductance of noncircular ducts, viz.,

$$S = 400 \frac{A^2}{OL} \text{ liters/sec} \quad (1)$$

where A = cross-sectional area in square inches

O = perimeter of duct in inches

L = length of duct in inches

(cf. Appendix) the correction factors K listed in Table 4.3, Appendix I, have not been applied.

4.1 PUMPING SPEED INSIDE DEE

Figure 4.1 shows the shape of the dee and illustrates the large deviation from the simple geometries often found in high vacuum systems. The back of the dee is to be a continuous aluminum channel beam running along its entire circumference, which will have a maximum of open area consistent with structural requirements. The rectangular tuning stub line, attached to the rear of the dee as shown, will have internal structural braces which will be perforated, as will also the sides of this stub line. The arrangements of these openings are shown schematically in Fig. 4.2. The philosophy of this arrangement is that, in addition to pumping the dee out through its mouth, there will be additional pumping speed, though admittedly small nevertheless valuable, gained through the openings along the back of the dee. Part of this additional speed comes from the perforations opening directly into the vacuum chamber proper and part by way of the tuning stub line and the openings in its sides. This additional pumping speed then can be considered as being in parallel with that through the opening at the mouth of the dee.

In order to simplify the calculations required, the conductance inside the dee is calculated, as for a rectangular duct, with its cross section taken in the XZ plane and its length along the Y axis. Further,

this is divided into two parts, (1) a tapered rectangular duct to which Eq. 1 can be applied directly, using average values of height and width of the duct, and (2) a circular section over which it is necessary to integrate the conductance for an incremental length ΔL , extending up to the point where the rectangular stub line is attached, at which point the dee effectively becomes a rectangular box. The orifices along the back of the dee are then taken into account separately in combining the conductance of the dee with the two parallel pumping paths mentioned above.

In the first section average height = 5.35 in., average width = 80 in., $A_{avg} = 430 \text{ in.}^2$, $O_{avg} = 160 \text{ in.}$, and $L = 16 \text{ in.}$ Thus,

$$S' = 400 \frac{(430)^2}{(160)(16)} = 29,000 \text{ liters/sec} \quad (2)$$

In the second section the conductance of an incremental length of duct ΔL is given by

$$S = 400 \frac{A^2}{O \cdot \Delta L} \quad (3)$$

and the total conductance of a number of these in series is given by

$$\frac{1}{S} = \frac{1}{\Delta S_1} + \frac{1}{\Delta S_2} + \dots \quad (4)$$

or

$$\frac{1}{S} = \frac{1}{400} \int_{L_1}^{L_2} \frac{O}{A^2} dL \quad (5)$$

From Fig. 4.1, then,

$$x^2 + y^2 = r^2 = (75)^2$$

so

$$x = \sqrt{r^2 - y^2}$$

While

$$A = 4x\bar{z}$$

where $\bar{z}$ is the average value of z , the half-height, over this section, and $\bar{z} = 2.2 \text{ in.}$

Now

$$O = 4(x + \bar{z}) = 4x + 8.8$$

$$A = 8.8x$$

$$dL = dy$$

(6)

Putting the expressions of Eq. 6 into Eq. 5 gives

$$\frac{1}{S''} = \frac{1}{400} \int_{L_1}^{L_2} \frac{4x + 8.8}{(8.8x)^2} dy \quad (6)$$

which, on substitution of the dependence of x on y and the limits of integration, yields the required integral,

$$\frac{1}{S''} = \frac{1}{400} \int_{19}^{68.5} \frac{4 \sqrt{r^2 - y^2} + 8.8}{77.7(r^2 - y^2)} dy$$

$$\frac{1}{S''} = \frac{1}{7760} \int_{19}^{68.5} \left(\frac{dy}{\sqrt{r^2 - y^2}} + \frac{2.2 dy}{r^2 - y^2} \right)$$

$$\frac{1}{S'} = k \left[\sin^{-1} \left(\frac{y}{r} \right) + \frac{2.2}{r} \tanh^{-1} \left(\frac{y}{r} \right) \right]_{-19}^{-68.5}$$

$$\begin{aligned} &= k \left\{ \left[\sin^{-1} \left(\frac{68.5}{75} \right) - \sin^{-1} \left(\frac{19}{75} \right) \right] + \frac{2.2}{75} \left[\tanh^{-1} \left(\frac{68.5}{75} \right) - \tanh^{-1} \left(\frac{19}{75} \right) \right] \right\} \\ &= k \left\{ \left[\sin^{-1} (.914) - \sin^{-1} (.254) \right] + \frac{2.2}{75} \left[\tanh^{-1} (.914) - \tanh^{-1} (.254) \right] \right\} \\ &= k \left[(1.154 - 0.263) + \frac{2.2}{75} (1.55 - 0.26) \right] \end{aligned}$$

$$\frac{1}{S''} = 0.930k = 1.2 \cdot 10^{-4}$$

From Eq. 2,

$$\frac{1}{S'} = 0.34 \cdot 10^{-4}$$

so that

$$S_{dee} = \frac{1}{\frac{1}{S'} \frac{1}{S''}} = \frac{1}{1.54 \cdot 10^{-4}} = 6500 \text{ liters/sec}$$

The fraction of the dee which opens into the rectangular tuning stub line at its rear is taken as the ratio of the planar angles $\theta/(\theta + \phi)$, as shown in Fig. 4.2, and the fraction of S_{dee} to be taken in series with the conductance of the stub line is then determined by this ratio. Now $\theta/(\theta + \phi) = 0.31$, so that

$$0.31S_{dee} = 2000 \text{ liters/sec} \quad (8)$$

4.2 PUMPING SPEED INSIDE TUNING LINE

The conductance of the rectangular tuning stub line is computed with the cross-sectional area A taken in the plane parallel to the Y axis and with the length L from the Y axis to the side of the stub line,

taking into account the 3-in.-diameter perforations in the single reinforcing brace in that half. This speed is then doubled for the two halves of this duct, giving S_{12} . Hereafter, the designation of the conductances S of the various branches of the vacuum circuit will have the definitions indicated by their subscripts in accordance with the equivalent circuit shown in Fig. 4.3; the term "resistance" is used there to indicate the reciprocal of the conductance. The use of resistance presents perhaps a clearer picture through the use of an electrical analogy.

With reference to Figs. 4.3 and 4.2, then, S_{12} is taken in series with S_{13} , $0.31S_{dee}$, and the 3 by 2 $\frac{3}{4}$ -in. orifices along the back of the dee opening into the stub line, i.e., S_{11} ; this compound conductance is denoted by S_C .

For S_{12} , $A = 252 \text{ in.}^2$, $O = 134 \text{ in.}$, $L = 30 \text{ in.}$, and

$$S'' = 400 \frac{(252)^2}{(134)(30)} = 6000 \text{ liters/sec}$$

The 3-in.-diameter holes have $A = 9(9\pi/4) = 20.2\pi$, so that

$$S'' = 75(20.2\pi) = 4750 \text{ liters/sec}$$

Thus

$$\frac{2}{S_{12}} = \frac{1}{S'} + \frac{1}{S''} = 3.77 \cdot 10^{-4}$$

$$S_{12} = 5300 \text{ liters/sec} \quad (9)$$

For S_{13} , $A = 18(3)(2.75) = 148$

$$S_{13} = 75(148) = 11,100 \text{ liters/sec} \quad (10)$$

For S_{11} , $A = 6(5)(2.75) = 83$

$$S_{11} = 75(83) = 6200 \text{ liters/sec} \quad (11)$$

Thus for S_C there results the value

$$S_C = 1/[(1/6200)(1/11,100)(1/5320)(1/2000)] = 1/(9.4 \cdot 10^{-4})$$

$$S_C = 1100 \text{ liters/sec} \quad (12)$$

4.3 PUMPING SPEED OF REGION SURROUNDING DEE

The conductance of the region of the vacuum chamber surrounding the dee was determined as follows: As is evident from Fig. 4.2, the cross-sectional area and the perimeter of what may be considered as the duct formed by two coaxial boxes varies in no regular way. Therefore, the conductance was approximated by subdividing it into a number of short, rectangular-shaped annular pumping lines with the cross-sectional area A parallel to the X axis and then adding these conductances in series (cf. Fig. 4.3). The sections into which it was found convenient to subdivide the vacuum chamber are indicated in Fig. 4.2 by section lines AA through PP. Now the section line GG passes through the center of the openings at the sides of the vacuum chamber to which the pump lines are attached. Therefore the conductance of that portion of the vacuum chamber between sections AA and GG, called S_A , is taken in parallel with that of the remainder, and the resulting net conductance taken in series with both pump lines at the sides of the vacuum chamber.

The outer skin of the dee is reinforced by means of hat-like structural members crossing it, as shown in Fig. 4.2, along the section lines AA, CC, EE, and GG. These constrictions are treated as orifices, large compared to the area surrounding them (cf. Eq. 44, Appendix II), whereas the regions between these section lines are taken to be the rectangular-shaped annular ducts with average values of cross-sectional area and perimeter taken as those values at the section lines BB, DD, and FF, respectively. In addition to this complication, the shape of the magnet shim which forms a major part of the top and bottom of the vacuum chamber effects a considerable variation in the parameters A and O. This is not shown in Fig. 4.2 but is taken into account in the following calculations, where the values of A, $\bar{A}$, O, and L have been taken from sectional views of a detailed drawing. A, O, and L are as previously defined, and $\bar{A}$ is the total cross-sectional area in square inches of the duct in which the orifice is situated. These values are listed in Table 4.1, along with those of the conductances and the sections to which they apply.

The net conductance S_A is then found from Eq. 45, Appendix II, giving

$$S_A = 37,300 \text{ liters/sec} \quad (12a)$$

Table 4.1 — Conductance for Sections AA to GG of Vacuum Chamber

Designation	Type of Element	A, in. ²	$\bar{A}$, in. ²	O, in.	L, in.	Conductance liters/sec
S ₁	Orifice	3352	3648			$3.1 \cdot 10^6$
S ₂	Duct from AA to CC	2182		622	16.4	$1.94 \cdot 10^5$
S ₃	Orifice	1208	1526			$4.35 \cdot 10^5$
S ₄	Duct from CC to EE	2068		676	18.2	$1.4 \cdot 10^5$
S ₅	Orifice	1600	1988			$6.15 \cdot 10^5$
S ₆	Duct from EE to GG	1776		536	18.2	$1.3 \cdot 10^5$
S ₇	Orifice	1156	1452			$4.26 \cdot 10^5$

The conductance to be taken in parallel with S_A will now be determined. Since at this stage the main concern is to determine the pressure which will exist inside the dee and in the region of the vacuum chamber in which particles are to be accelerated, the conductance S_C previously found is taken in series with that of the rectangular-shaped annulus about the tuning stub line and dee between sections KK and GG. That is, if we call this combined conductance S_D ,

$$\frac{1}{S_D} = \frac{1}{S_C} + \frac{1}{S_{10}} + \frac{1}{S_9} \quad (13)$$

It might be mentioned at this point that the 20-in. diffusion pump at the rear of the condenser box has been added mainly to provide sufficient pumping speed in the region of the rotating condenser to ensure the required vacuum between the condenser blades. This pump will at the same time provide some additional pumping capacity for the vacuum chamber as a whole, and this is determined by taking the conductance of the entire vacuum chamber in series with the 20-in. pump; for the present this will not be included, since it will not affect the evaluation of the size of the diffusion pumps to be included in the pumping leads to the sides of the vacuum chamber.

For S_{10} , $A = 1624 \text{ in.}^2$, $O = 348 \text{ in.}$, $L = 33 \text{ in.}$,

$$S_{10} = 400 \frac{(1624)^2}{(348)(33)} = 92,600 \text{ liters/sec} \quad (14)$$

For S_9 , $A = 1308 \text{ in.}^2$, $O = 380 \text{ in.}$, $L = 11 \text{ in.}$,

$$S_9 = 400 \frac{(1308)^2}{(380)(11)} = 175,000 \quad (15)$$

Thus,

$$\frac{1}{S_D} = \frac{1}{1100} + \frac{1}{92,600} + \frac{1}{175,000} = 92.8 \cdot 10^{-5}$$

$$S_D = 1080 \text{ liters/sec}$$

4.4 NET PUMPING SPEED OF VACUUM CHAMBER WITH RESPECT TO SIDE OUTLETS

The net conductance of the vacuum chamber, with respect to the side outlets of the chamber S_{VC} , is now found as the combined conductance of S_A , S_D , and that fraction of S_{dee} opening into the vacuum chamber between sections AA and JJ, all taken in parallel. This latter conductance will be the sum of $0.69S_{dee}$ and S_8 taken in series. Therefore,

$$S_8 = 75(121) = 9070 \text{ liters/sec} \quad (17)$$

so that

$$\frac{1}{(0.69S_{dee} + S_8)} = \frac{1}{4500} + \frac{1}{9070} = 3.33 \cdot 10^{-4}$$

and

$$(0.69S_{dee} + S_8) = 3000 \text{ liters/sec} \quad (18)$$

Then,

$$S_{VC} = S_A + S_D + (0.69S_{dee} + S_8)$$

$$37,300 + 1080 + 3000 = 41,400 \text{ liters/sec} \quad (19)$$

S_{VC} is the conductance, as specified above, to the region of the vacuum chamber at the mouth of the dee, considering the two parallel pumping paths through the dee and about the dee. In order to get the pumping speed with respect to the region inside the dee, again as seen from the side outlets of the vacuum chamber, it is necessary to include the pumping speed of the dee itself through its mouth in series with S_A . That is, the dee will be pumped out through its mouth as well as through the openings along its back; the latter has already been included in S_D and in $(0.69S_{dee} + S_8)$. This conductance, which will be designated by S_{VC}' , is as follows:

$$\begin{aligned} S_{VC}' &= \frac{1}{\frac{1}{S_{dee}} + \frac{1}{S_A}} + S_D + (0.69S_{dee} + S_8) \\ &= \frac{1}{\frac{1}{6500} + \frac{1}{37,300}} + 1080 + 3000 = 9580 \text{ liters/sec} \end{aligned} \quad (20)$$

4.5 PUMPING SPEED OF PUMP LEADS AT SIDE OUTLETS

Consider the pumping speed of one side arm taken in two sections in each of which the length L is taken along the center line.

For the first section, $A = (56)(20) = 1120 \text{ in.}^2$, $O = 2(56 + 20) = 152 \text{ in.}$, $L = 24 \text{ in.}$, and

$$S_1 = 400 \frac{(1120)^2}{(152)(24)} = 137,000 \text{ liters/sec}$$

For the second section, $A = (50)(20) = 1000 \text{ in.}^2$, $O = 2(50 + 20) = 140 \text{ in.}$, $L = 72 \text{ in.}$, and

$$S_2 = 400 \frac{10^6}{(140)(72)} = 39,700 \text{ liters/sec}$$

Then for both pump leads in parallel,

$$\frac{2}{S_L} = \frac{1}{S_1} + \frac{1}{S_2} = 3.25 \cdot 10^{-5}$$

and

$$S_L = 61,600 \text{ liters/sec} \quad (21)$$

4.6 EFFECTIVE PUMPING SPEED WITH RESPECT TO PUMPS IN SIDE ARMS

By combining S_L plus S_{VC} and S_L plus S_{VC}' , the effective pumping speed is obtained for the region of the vacuum chamber at the mouth of the dee and inside the dee, respectively, with respect to the mouth of the diffusion pumps to be located in the two side arms, as shown in Fig. 4.2. These two effective conductances are designated S_{E1} and S_{E1}' , respectively. Then,

$$S_{E1} = \frac{1}{(1/S_L + 1/S_{VC})} = \frac{1}{(4.04 \cdot 10^{-5})} = 24,800 \text{ liters/sec} \quad (22)$$

$$S_{E1}' = \frac{1}{(1/S_L + 1/S_{VC}')} = \frac{1}{(1.2 \cdot 10^{-4})} = 8300 \text{ liters/sec} \quad (23)$$

It should be mentioned that in obtaining these values the conductance of that half of the vacuum chamber not occupied by the dee has been neglected. The pumping speed of that half of the vacuum chamber is approximately 200,000 liters/sec and is in the ratio of 5.4:1 with S_A , with which it would be added in series; its inclusion would therefore reduce the speed S_{VC} by less than 17 per cent with respect to the farthest point of that region of the vacuum chamber. This reduction is, of course, small compared to that inside the dee and has therefore been neglected.

4.7 COMPARISON OF 20- AND 32-IN. DIFFUSION PUMPS IN SIDE DUCTS

From the experience of other cyclotron installations it is apparent that at least 20-in. diffusion pumps will be required in the side pump lines, and there remains some question as to whether it would not be advisable to use 32-in. diffusion pumps. This is especially so, since the amount of gas resulting from outgassing of the large exposed surface area is a matter of conjecture. Thus the system should be designed to attain the required vacuum, while handling the required flow of gas from the ion source, with sufficient excess pumping capacity available to provide for the virtual leaks to be expected from outgassing. Also in a system as large as this, it is unlikely that a completely vacuum-tight assembly may be

realized, so that there should be some allowance for additional pumping capacity to handle a small amount of inleakage.

20-in. Diffusion Pumps

From the performance curve, as furnished by the manufacturer, for a Distillation Products MC-7000 20-in. diffusion pump with an FB-01 Freon baffle and Myvane-20 oil, the following values of speed vs. pressure for air were obtained.

P_B	$5 \cdot 10^{-5}$	$3 \cdot 10^{-5}$	$2 \cdot 10^{-5}$	10^{-5}	$9 \cdot 10^{-6}$	mm Hg
S_{PB}	3500	3500	3450	3410	3400	liters/sec
$2S_{PB}$	7000	7000	6900	6820	6800	liters/sec
$Q (= P_B \cdot 2S_{PB})$	0.35	0.21	0.138	0.068	0.061	liter-mm/sec

With no sources or sinks the same quantity of gas, Q , should flow through all series elements of a circuit, so that the following equations result.

$$Q = P_B \cdot 2S_{PB} = S_{E1}(P - P_B) = S_{net}P \quad (24)$$

where

Q = rate of flow of gas, liter-mm/sec

P_B = pressure at throat of pump, mm Hg

S_{PB} = speed of pump and baffle combined, liters/sec

P = pressure at point in system for which S_{E1} is defined in mm. Hg

S_{net} = net speed of over-all system (pump, baffle, and lines), liters/sec = $1/(1/2S_{PB} + S_{E1})$

From Eq. 24, $P = Q/S_{E1} + P_B$. The pressure P , which would be obtained in the region of the vacuum chamber near the mouth of the dee, is
for $P_B = 10^{-5}$,

$$P = 0.068/24,800 + 10^{-5} = (0.27 + 1) \cdot 10^{-5} = 1.27 \cdot 10^{-5} \text{ mm Hg}$$

for $P_B = 9 \cdot 10^{-6}$,

$$P = 0.061/24,800 + 0.9 \cdot 10^{-5} = (0.25 + 0.9) \cdot 10^{-5} = 1.15 \cdot 10^{-5} \text{ mm Hg}$$

If 200 ml/hr is considered as a maximum rate of gas injection, this results in 4220 liters/sec at 10^{-5} mm Hg, or 0.042 liter-mm/sec. Although this gas will be hydrogen or deuterium, for which the effective pumping speed of the vacuum system will be higher than for air, there will be very little difference in the pumping speed of the pump itself at these low pressures. Thus at a pressure of $1.15 \cdot 10^{-5}$ mm Hg in the vacuum chamber, it can be seen, from the preceding conclusions, that with 20-in. diffusion pumps there is little excess pumping capacity available.

32-in. Diffusion Pumps

From the performance curve, as furnished by the manufacturer, for a Distillation Products MC-10000 32-in. diffusion pump with an FB-02 Freon-cooled baffle and Myvane-20 pump oil, the following values of speed vs. pressure for air were obtained.

P_B	10^{-5}	$9 \cdot 10^{-6}$	$8 \cdot 10^{-6}$	$7 \cdot 10^{-6}$	$6 \cdot 10^{-6}$	mm Hg
S_{PB}	6,800	6,800	6,700	6,600	6,500	liters/sec
$2S_{PB}$	13,700	13,600	13,400	13,200	13,000	liters/sec
$Q(= P_B \cdot 2S_{PB})$	0.137	0.122	0.107	0.0925	0.078	liter-mm/sec

From Eq. 24, $P = Q/S_{E1} + P_B$. Using the values for Q corresponding to the various pressures at the throat of the pump P_B , values of P are obtained that may be compared with those due to a 20-in. diffusion pump; this also permits a comparison of the values of Q . Thus for $P_B = 10^{-5}$,

$$P = 0.137/24,800 + P_B = 1.55 \cdot 10^{-5} \text{ mm Hg}$$

for $P_B = 8 \cdot 10^{-6}$,

$$P = 0.107/24,800 + P_B = 1.23 \cdot 10^{-5} \text{ mm Hg}$$

for $P_B = 7 \cdot 10^{-6}$,

$$P = 0.0925/24,800 + P_B = 1.07 \cdot 10^{-5} \text{ mm Hg}$$

On the basis of the pressure which could be obtained with 20-in. pumps, i.e., $P = 1.15 \cdot 10^{-5}$, and $Q = 0.061$ liter-mm/sec, 32-in. pumps will give a value of $Q = 0.10$ (approximately), at the same pressure. Thus the 32-in. pumps will handle almost twice as much gas as the 20-in. pumps, and the decision to use 32-in. pumps is an obvious one. To this pumping capacity, of course, will be added that available through the 20-in. diffusion pump at the rear of the condenser box.

4.8 PUMPING SPEED AVAILABLE THROUGH CONDENSER BOX

It is desirable to include now the contribution of the 20-in. diffusion pump located at the rear of the condenser box to the over-all pumping speed, both with respect to the region of the vacuum chamber near the mouth of the dee and to the inside of the dee itself. The net conductance of the vacuum chamber and condenser box taken up to the throat of the pump will, for the two cases mentioned, be denoted by S_{E2} and $S_{E2'}$, respectively. For the present purpose the region occupied by the rotating condenser and stator blades is taken as being completely filled by them, except for the three parallel pumping channels, as shown in Fig. 4.2.

The conductance of the annular space between sections AA and FF is now taken in parallel with that through the side openings of the dee; this is designated S_a . Then (cf. Figs. 4.2 and 4.3)

$$S_a = \frac{1}{\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \frac{1}{S_4} + \frac{1}{S_5} + \frac{1}{2S_6}} + \frac{1}{\frac{1}{S_8} + \frac{1}{0.69S_{dee}}} \\ S_a = 49,000 + 3000 = 52,000 \text{ liters/sec} \quad (25)$$

This conductance will be in series with that of the annulus between sections FF and KK. Designating this combined conductance by S_b , then

$$S_b = \frac{1}{\frac{1}{S_a} + \frac{1}{2S_6} + \frac{1}{S_7} + \frac{1}{S_9} + \frac{1}{S_{10}}} = 23,800 \text{ liters/sec} \quad (26)$$

The conductances S_b and S_c will be in parallel, and their combined conductance will be in series with combined conductance of S_{14} , S_{17} , and S_{19} , all taken in series.

Thus

$$S_{E2} = \frac{1}{\frac{1}{S_b + S_C} + \frac{1}{S_{14}} + \frac{1}{S_{17}} + \frac{1}{S_{19}}}$$

S₁₇ is made up of the three parallel ducts, the outer two being identical, for which the speed is computed as for a long duct, and the end correction is introduced as an orifice correction.

For the center duct, A = 210 in.², O = 61 in., L = 23 in.,

$$S_{\text{orifice}} = 75(210) = 15,800 \text{ liters/sec}$$

$$S_{\text{duct}} = 400(210)^2/(61)(23) = 12,600 \text{ liters/sec}$$

$$S' = 1/(1/15,800 + 1/12,600) = 7000 \text{ liters/sec}$$

For the outer ducts, A = 240 in.², O = 64 in., L = 23 in.,

$$S_{\text{orifice}} = 75(240) = 18,000 \text{ liters/sec}$$

$$S_{\text{duct}} = 400(240)^2/(64)(23) = 15,700 \text{ liters/sec}$$

This is for one of the ducts. Therefore the combined speed of both of them is

$$S'' = 2/(1/18,000 + 1/15,700) = 16,800 \text{ liters/sec}$$

Then

$$S_{17} = S' + S'' = 16,800 + 7000 = 23,800 \text{ liters/sec} \quad (28)$$

For S₁₉, A = 1680 in.², O = 208 in., L = 15 in.,

$$S_{19} = 400(1680)^2/(208)(15) = 362,000 \text{ liters/sec} \quad (29)$$

Putting the values from Eqs. 28 and 29 and that previously obtained for S₁₀ (which is the same as S₁₄) in place of S₁₄ in Eq. 27,

$$S_{E2} = 10,500 \text{ liters/sec} \quad (30)$$

The only change required to obtain S_{E2'} is to include in series with the conductance between sections AA and FF that of the dee itself. Then

$$S_{a'} = \frac{1}{\frac{1}{S_{\text{dee}}} + \frac{1}{S_1} + \dots + \frac{1}{S_5} + \frac{1}{2S_6}} + \frac{1}{\frac{1}{0.69S_{\text{dee}}} + \frac{1}{S_8}}$$

$$S_{a'} = 8740 \text{ liters/sec}$$

Using this value of S_{a'} in place of S_a in Eq. 26 get S_{b'}, which is to be put into Eq. 27 in place of S_b. Finally,

$$S_{E2} = 4750 \text{ liters/sec} \quad (31)$$

The preceding values of conductance have all been for air. The conductance for hydrogen will be greater (cf. Eq. 40, Appendix I) by the factor $\sqrt{M_{\text{air}}/M_{\text{H}_2}} = \sqrt{29/2} = 3.8$. In summation, the values of the effective pumping speeds of the system are tabulated for both air and hydrogen.

	SE1	SE1'	SE2	SE2'
Air, liters/sec	24,800	8,300	10,500	4,750
Hydrogen, liters/sec	94,000	31,500	40,000	18,000

4.9 NET SPEED AND PUMP-DOWN TIME

With the arrangement of the three diffusion pumps as shown in Fig. 4.2 and the effective pumping speeds listed above, it is now possible to obtain a curve for the net speed of the combination in the high-vacuum region. This is accomplished by first computing a net speed curve for the two 32-in. diffusion pumps combined with S_{E1} in series, and for the 20-in. diffusion pump combined with S_{E2} in series, in each case plotted against the pressure in the vacuum chamber. The relations required for finding the net speed and the pressure in the vacuum chamber are given by Eq. 24. From these two curves of net speeds vs. pressure in the vacuum chamber, the combined net-speed curve is obtained by adding the individual net speeds at corresponding values of pressure in the vacuum chamber. The curve obtained in this way is shown in Fig. 4.4.

Figure 4.6 shows the final arrangement and choice of vacuum pumps. Each of the diffusion pumps is backed by a Distillation Products MB-300-06 booster pump and a Kinney DVD-8810 mechanical pump, and each forms an independent parallel branch, since the valves in the cross connections will normally be closed. The speed curves calculated for the three diffusion pumps in parallel, the three booster pumps in parallel, and the three mechanical pumps in parallel are shown in Fig. 4.5, where the speed is given in liters per second corrected to a standard pressure of 760 mm Hg. Figure 4.5 also shows the speed curve of the Kinney DVD-14918 mechanical roughing pump. Although the abscissa is indicated as the pressure in the vacuum chamber for all these curves, this pressure applies exactly only for the diffusion pumps. For the mechanical roughing pump it applies closely throughout the range of pressure of interest, since the pressure drop in the large roughing line and vacuum chamber is very small, approximately 2 per cent at 50 μ . However, for the booster pump, the speed is plotted against the pressure at the inlet, and this is certainly different from that in the vacuum chamber in the range of pressure where only the booster pump is carrying the load. This can be approximately corrected in computing the pump-down time as follows: The speed of the diffusion pumps decreases rapidly beyond the knee of the curve, as is seen in Fig. 4.4, to a value determined by the booster pump. If in this breakdown region the speed of the diffusion and booster pump combination is considered to be that of the booster pump alone and if this is combined in series with the effective conductance of the vacuum chamber as previously determined, an approximate but somewhat too high value of the net speed in this pressure range is obtained. The conductance as found for free molecular flow will actually be too low a value for this higher pressure range, but this will tend to compensate for the assumed speed.

The total volume of the vacuum system is 15,000 liters. Neglecting outgassing and assuming a perfectly tight system, the pump-down times are as follows:

1. Roughing cycle, time required to pump down from 760 mm Hg to 45 μ , with the aid of the Kinney DVD-14918, 30 min
2. Finishing cycle, time required to pump down from 45 μ to $4 \cdot 10^{-6}$ mm Hg, 3 min
3. Total pump-down time from 760 mm to $4 \cdot 10^{-6}$ mm Hg, 33 min
4. Recovery time for a gas burst of such magnitude as to bring the pressure in the vacuum chamber from $4 \cdot 10^{-6}$ to 10^{-3} mm Hg, 8 sec

The pressure of $4 \cdot 10^{-6}$ mm Hg represents the approximate pressure which can be obtained in the vacuum chamber with a gas injection of 200 ml/hr or 0.042 liter-mm/sec. The speed of the diffusion pumps is not accurately known for hydrogen or deuterium at these low pressures, but it will be only slightly greater than that for air, and this value of limiting pressure is based on the pump speeds for air. The fact that the conductances are considerably higher for hydrogen, therefore, will have very little effect toward increasing the net speed in this region.

4.10 Description of Vacuum-System Components and General Layout

Figure 4.6 shows a schematic arrangement of the vacuum system components, whereas Fig. 4.2 shows more specifically the points of attachment of the diffusion pumps. Each of the diffusion pumps is connected to the underside of an extension of the vacuum chamber through the intermediary of a flange-connected Freon-cooled baffle. The extension of the vacuum chamber consists of the two side ducts in

the case of the 32-in. pumps and the condenser box in the case of the 20-in. pump. Each diffusion pump is preceded by a pneumatically operated high-vacuum valve, which has its sealing disk located in the extensions of the vacuum chamber directly above the diffusion pump and seals, against the Freon baffle flange.

The fore-outlets of the diffusion pumps are connected directly to 8-in. booster pumps through 8-in. steel tubing containing a 6-in. manual high-vacuum valve. These lines are cross-connected by similar lines containing 6-in. manual high-vacuum valves. These cross-connections are normally closed, so that the three pumping channels operate as independent parallel channels. They permit, however, the backing of all three diffusion pumps by a combination of any two of the booster pumps, should any one of the booster or mechanical backing pumps become inoperative, merely by closing off the valve in the booster line and opening the cross-connecting valves. In this way full high-vacuum operation may still be obtained. The booster pumps are each backed by an individual mechanical roughing pump through 4-in. steel tubing containing a 4-in. manual high-vacuum valve. This line also contains a bleeder connection through a $\frac{3}{4}$ -in. solenoid-operated high-vacuum valve which will be opened automatically when the mechanical pump is shut down so as to break the vacuum in the line and prevent the pump oil from backing up the line from the mechanical pump.

The large mechanical roughing pump is connected directly to the vacuum chamber proper through 8-in. steel tubing attached at a convenient location on one of the side pumping ducts. This line contains an 8-in. electrically operated, automatic high-vacuum valve as well as the bleeder connection described above.

The supports for the diffusion pumps will be adjustable over a short distance, so that the diffusion-pump flanges may be drawn up tightly against their gaskets. Thus the piping connections to these pumps will have a flexible metal connection to permit the piping to be permanently fixed in place while permitting the previously mentioned adjustment. Similarly, the mechanical pumps will be connected to their pumping lines through a flexible metal connection to prevent transmission of vibrations from the pumps through the piping. All flexible connections will be made in a vertical run of piping in order to prevent any possible accumulation of oil within them.

The 32-in. diffusion pumps, Distillation Products MC-10000 oil diffusion pumps with 208-volt heaters, require a heater input variable from 5 to 7 kw. A forepressure of 60 μ is required for their operation. Two are required.

The 20-in. diffusion pump consists of a Westinghouse casing modified with a Distillation Products 20-in. jet and 208-volt heaters to have the operating characteristics of the Distillation Products MC-7000. Heater input required is from 3.5 to 4.5 kw. Forepressure required is 50 μ . One is required.

The 32-in. baffles, Distillation Products FB-02 32-in. Freon-cooled baffles, designed so that the baffle flange will serve as the valve seat. Two are required.

The 20-in. baffle, Distillation Products FB-01 20-in. Freon-cooled baffle, also designed so that the baffle flange serves as the valve seat. One is required.

The 32- and 20-in. high-vacuum valves are pneumatically operated valves designed by the Nuclear Research Center engineering staff. Each is operated by two air pistons through a linkage arrangement and designed to lock the valve in either the fully open or closed position. They will close automatically and swiftly at a predetermined pressure and lock closed, so as to hold shut even against atmospheric pressure. Two 32-in. and one 20-in. valves are required.

The booster pumps, Distillation Products MB-300-06 booster pumps with 208-volt heaters, require a heater input of from 1.5 to 2.5 kw. Forepressure required is 160 μ . Three are required.

The mechanical backing pumps, Kinney DVD-8810, have 208-volt 5-hp motors and 208-volt solenoid valves in the oil line. Three are required.

The mechanical roughing pump, Kinney DVD-14918, has a 208-volt 15-hp high-starting-torque motor for automatic operation of the pump and a 208-volt solenoid valve in the oil line. One is required.

Five 6-in. manual Chapman high-vacuum valves, three 4-in. manual Distillation Products QSV-4 high-vacuum valves, four $\frac{3}{4}$ -in. solenoid-operated Distillation Products SV-02A high-vacuum valves, one 1-in. manual Distillation Products KVS-1 high-vacuum valve, and one 8-in. motor-operated Distillation Products QSV-8 high-vacuum valve are required.

A Freon compressor unit of 1-ton capacity will be used to supply the Freon baffles with refrigerant. It is estimated that a $\frac{1}{2}$ -ton capacity would be ample to handle all three baffles, so that there will be excess capacity available for any future additions, such as Freon-cooled baffles at the mouth of the MB-300 booster pumps, if they should be required.

All vacuum lines are to be seamless steel tubing with all direct connections to be welded. Flanged connections are to be made up of welding neck flanges with double O-ring gasket grooves machined in a standard flat face flange and provided with a pump-out connection between the two gaskets.

4.11 OPERATING PROCEDURE

Starting up the vacuum system initially, or a complete shutdown, necessitates the following sequence of operations, most of which will be automatic.

1. Turn on the cooling water to the diffusion, booster, and mechanical pumps. This cooling water is supplied directly from the house water system, rather than from the recirculating-water system, so that the temperature of the water will be approximately 60° F. This cooling water is supplied from a common header which branches out into parallel lines to feed the individual pumps. In the individual return lines from each pump, there will be located a flow switch which will be interlocked with the power supply to the heaters or motors, respectively, so that they may not be turned on without cooling water flowing, or so that they will shut off if the cooling water ceases to flow. In addition thermostats will be used in the cooling-water lines from the mechanical pumps, in order to turn these pumps off, if the temperature becomes excessive. Similarly, thermostats will be attached to the cooling-water lines around the diffusion- and booster-pump casings at a point above the heaters, in order to shut them off, if the temperature becomes excessive. The action taken by the various interlocks will, of course, be indicated on the control panel.

2. The 6-in. manual valves in the cross-connections between the booster pumps should be closed, and the 6- and 4-in. manual valves in each of the three parallel pumping channels should be opened. The air-pressure supply to the pneumatic valves should be on, and these three valves should be closed.

3. All four mechanical pumps are turned on. The solenoid bleeder valves, which are normally open, close as soon as power is supplied to the pumps. The motor-operated valve in the roughing line opens when the roughing pump is turned on (so long as the pressure on either side of the valve is approximately equal), and the vacuum chamber is now being roughed out. In the meantime the smaller mechanical pumps are roughing out the three parallel high-vacuum pumping channels up to the closed, pneumatically operated valves.

4. Simultaneously with step 3 the Freon compressor is turned on. Turning the compressor on at this stage will allow the Freon baffles to become sufficiently cold by the time the diffusion pumps are heated up. The power supply to the mechanical pumps will be interlocked with that of the compressor motor.

5. When the forepressure of the diffusion pumps reaches 50 μ , the heaters to the booster and diffusion pumps are turned on.

6. After approximately 30 min, the roughing pumps will have brought the pressure in the vacuum chamber down to 45 μ . At this pressure in the vacuum chamber the motor-operated valve in the roughing line will close, and the roughing pump will shut down, its bleeder valve opening as the pump shuts off.

7. Simultaneously with step 6 the pneumatically operated 32 and 20-in. valves will open, if the pressure on the diffusion pump sides of these valves is below 50 μ , permitting the three pumping channels to continue pumping down the vacuum chamber which will attain its operating pressure of approximately $4 \cdot 10^{-6}$ mm Hg in several minutes, assuming no outgassing and no leaks. Actually, of course, some indeterminately longer time will be required due to the outgassing which will take place, especially at the initial pump down or after the chamber has been open to the air for some time.

4.12 AUTOMATIC FAULT SEQUENCE

If a leak should occur in the vacuum system, the vacuum components will automatically follow a sequence of shutdown, depending on where the leak occurs. If the leak should be in one of the parallel pumping channels, the pneumatically operated valve will close off this channel from the vacuum chamber when the forepressure of the diffusion pump reaches 50μ . Also the heaters on both the booster and diffusion pumps in this channel will be turned off when the forepressure of the diffusion pump reaches 50μ . If the leak is on the exhaust side of the booster pump, this pump will very shortly become blocked (at a forepressure 160μ) and raise the forepressure of the diffusion pump to that value which shuts off both their heaters and the pneumatically operated valve, as mentioned above. Full high-vacuum pumping capacity may then be restored at will by shutting off the 6-in. manual valve in the line connecting the booster and the diffusion pump, if the leak has occurred on the booster- or mechanical-pump side of this valve, and by opening the cross-connecting valves. The diffusion pump will then automatically return into operation, when its forepressure reaches the value required to turn on the heaters and open the pneumatically operated valve.

If the leak has occurred on the diffusion side of the 6-in. manual valve in the diffusion-to-booster pump connection, it will be necessary to keep this channel out of service until it is convenient to search out the leak and repair it. The leak can be localized to either side of the valve simply by closing it and manually shorting out the vacuum-pressure interlocks, so as to be able to manually return the mechanical backing pump and the booster pump into operation. The ultimate pressure obtained at the outlet of the booster pump, which should be 10^{-4} mm Hg, or lower, and which should be attained in a very short time, will indicate whether or not the leak is in that part of the circuit.

If a leak should occur in the vacuum chamber, all three pneumatically operated valves will close when the pressure in the vacuum chamber reaches 45μ , but the three pumping channels will continue to operate with these valves closed. When the pressure in the vacuum chamber reaches 45μ , the mechanical roughing pump will automatically be turned on, pumping down the roughing line up to the motor-operated valve in its line. When the pressure on both sides of this valve becomes approximately equal, this valve will open allowing the roughing pump to pump out the vacuum chamber. If this leak was in actuality only a gas burst (which would be a tremendous gas burst to raise the pressure to this extent) and the vacuum chamber can be roughed down to 45μ within 20 min time from the start of the roughing pump, the diffusion pumps will again be opened up to the chamber and continue their normal operation. If, on the other hand, the roughing pump cannot pump down the vacuum chamber in the 20 min time allotted, a time-delay circuit will shut off the roughing pump circuit only. The three parallel pumping channels, which have been shut off from the vacuum chamber, continue to operate in either of the above cases.

APPENDIX I

CONDUCTANCE OF DUCTS OF NONCIRCULAR CROSS SECTION

The rate of flow of gas in the region of free molecular flow for a duct of variable cross section was given by Knudsen¹ as

$$Q = \frac{1}{\sqrt{\rho_1}} \frac{p_1 - p_2}{W} \quad (32)$$

W is a parameter of the form and dimensions of the duct only and $W \sqrt{\rho_1}$ may be considered as the "resistance" of the duct to the flow of the gas.

$$W = \frac{3}{8} \sqrt{\frac{\pi}{2}} \int_0^L \frac{O}{A^2} dl \quad (33)$$

where O = perimeter of the duct

A = cross-sectional area of the duct

ρ_1 = ratio of density to pressure of the gas, ρ/p

$p_1 - p_2$ = pressure drop across the length L of the duct

Q = rate of gas flow in consistent units

Since for a perfect gas $pV = nRT$ and $n = w/M$, $pV = wRT/M$,

where V = total volume of gas

n = number of moles of gas in that volume

R = universal gas constant

T = absolute temperature

w = mass of gas in volume V

M = molecular weight of gas

Since $\rho = w/V$, ρ_1 can be written as follows:

$$\rho_1 = \frac{\rho}{p} = \frac{w}{pV} = \frac{M}{RT} \quad (34)$$

Putting the value for ρ_1 into Eq. 32 results in

$$Q = \sqrt{\frac{RT}{M}} \frac{p_1 - p_2}{W} \quad (35)$$

For ducts of constant cross section throughout the length L , Eq. 32 is directly integrable, giving

$$W = \frac{3}{8} \sqrt{\frac{\pi}{2}} \frac{OL}{A^2} \quad (36)$$

which, when used in Eq. 35, gives

$$Q = \frac{8}{3} \sqrt{\frac{2}{\pi}} \sqrt{\frac{RT}{M}} \frac{A^2}{OL} (p_1 - p_2) \quad (37)$$

Equation 37 may then be written in the more familiar form

$$Q = S(p_1 - p_2) \quad (38)$$

where S is the "conductance" or "speed" of the tube, and is given by

$$S = \frac{8}{3} \sqrt{\frac{2}{\pi}} \sqrt{\frac{RT}{M}} \frac{A^2}{OL} \quad (39)$$

A formula for the conductance of a circular duct can be obtained directly from Eq. 39 by putting

$$\frac{A^2}{O} = \left(\frac{\pi D^2}{4} \right)^2 \frac{1}{\pi D} = \frac{\pi D^3}{16}$$

Then Eq. 39 becomes

$$S = \frac{1}{6} \sqrt{2\pi} \sqrt{\frac{RT}{M}} \frac{D^3}{L}$$

Now $R = 8.312 \times 10^7$ ergs/mole, °K, so that

$$S = \frac{1}{6} \sqrt{2\pi} \sqrt{\frac{8.312 \times 10^7 T}{M}} \frac{D^3}{L}$$

or rewriting this as

$$S = \frac{1}{6} \sqrt{2\pi} \sqrt{\frac{8.312 \times 10^7 (273)T}{273M}} \frac{D^3}{L}$$

so that

$$S = \frac{1}{6} (2.5) (15.1 \times 10^4) \frac{T}{273M} \frac{D^3}{L}$$

or, finally,

$$S = 6.3 \times 10^4 \sqrt{\frac{T}{273M}} \frac{D^3}{L} \text{ cm}^3/\text{sec} \quad (40)$$

In this form Eq. 40 agrees, for example, with the formula for the pumping speed of a circular tube, neglecting the end effects of the tube, as given by Shoupp.²

Equation 39 can be expressed in a more directly applicable form, as follows: If the gas is assumed to be air, $M = 29$, and if a reasonable and convenient value is chosen for the temperature of the air, $T = 300^\circ\text{K}$, then

$$S = \frac{8}{3} \sqrt{\frac{2}{\pi}} \sqrt{\frac{8.312 \times 10^7 (300)}{29}} \frac{A^2}{OL}$$

or

$$S = 6.25 \times 10^4 \frac{A^2}{OL} \text{ cm}^3/\text{sec}$$

By converting this in terms of liters and changing the dimensions in which A, O, and L are to be measured from centimeters to inches gives

$$S = (62.5) (2.54) \frac{A^2}{OL} \text{ liters/sec}$$

Finally, then*

$$S = 403 \frac{A^2}{OL} \text{ liters/sec} \quad (41)$$

where A = cross-sectional area of duct, in.²

O = perimeter of duct, in.

L = length of duct, in.

*For convenience of calculation the value 400 is used rather than 403; the results are hardly affected since the formula is only an approximation.

It must be emphasized that Eq. 41 is directly applicable to ducts of approximately circular cross section, and that, when applied to ducts of other than circular cross section, sizable errors may be introduced as indicated in Table I below. In detailed calculations Clausing⁴ deduced more accurate values for the conductance, in comparison with which those values resulting from Eq. 41, for various ratios of width a to height b for a rectangular duct, result in the percentage errors listed in Table I.

Table 4.2—Percentage Error in Conductance As Calculated from Eq. 41

a/b	1	2	3
Percentage error	-11	-15	-20

Values³ of the correction factor K, by which the right side of Eq. 41 must be multiplied to give a more accurate value of conductance, are listed in Table 4.3.

Table 4.3—Correction Factor K for Eq. 41

a/b	1	1.5	2	3	5	8	10
K	1.108	1.126	1.151	1.198	1.297	1.400	1.444

REFERENCES

1. L. B. Loeb, "The Kinetic Theory of Gases," 2d ed., p. 292, McGraw-Hill Book Company, Inc., New York, 1934.
2. W. E. Shoupp, Westinghouse Research Report R-94318-B, Jan. 6, 1943, p. 3.
3. C. E. Normand, Design of High Vacuum Systems, Ind. Eng. Chem., 40: 785 (1948).

APPENDIX II

SUMMARY OF CONDUCTANCE FORMULAS

Free-Molecular-Flow Region:

For long ducts of arbitrary cross section, neglecting end effects,

$$S = 400 \frac{A^2}{OL} \quad (42)$$

For an orifice small relative to the area surrounding it,

$$S = 75A \quad (43)$$

For an orifice large relative to the area surrounding it,

$$S = 75A \frac{\bar{A}}{\bar{A} - A} \quad (44)$$

For conductances in series,

$$\frac{1}{S_{\text{total}}} = \frac{1}{S_1} + \frac{1}{S_2} + \cdots \quad (45)$$

For Conductances in parallel,

$$S_{\text{total}} = S_1 + S_2 \cdots \quad (46)$$

In all the above formulas

A = free cross-sectional area of duct or orifice, in.²

A = total cross-sectional area of duct, in.² in which orifice is situated

O = perimeter of duct, in.

L = length of duct, in.

S = pumping speed, liters/sec

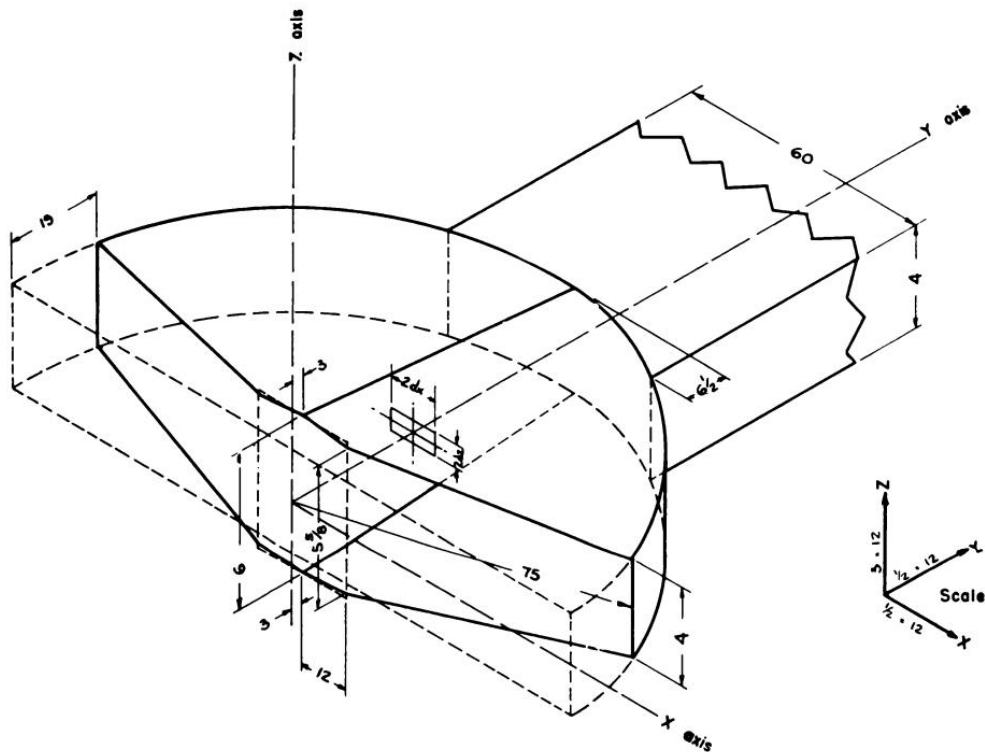


Fig. 4.1—Dee and stub line. (All dimensions are given in inches.)

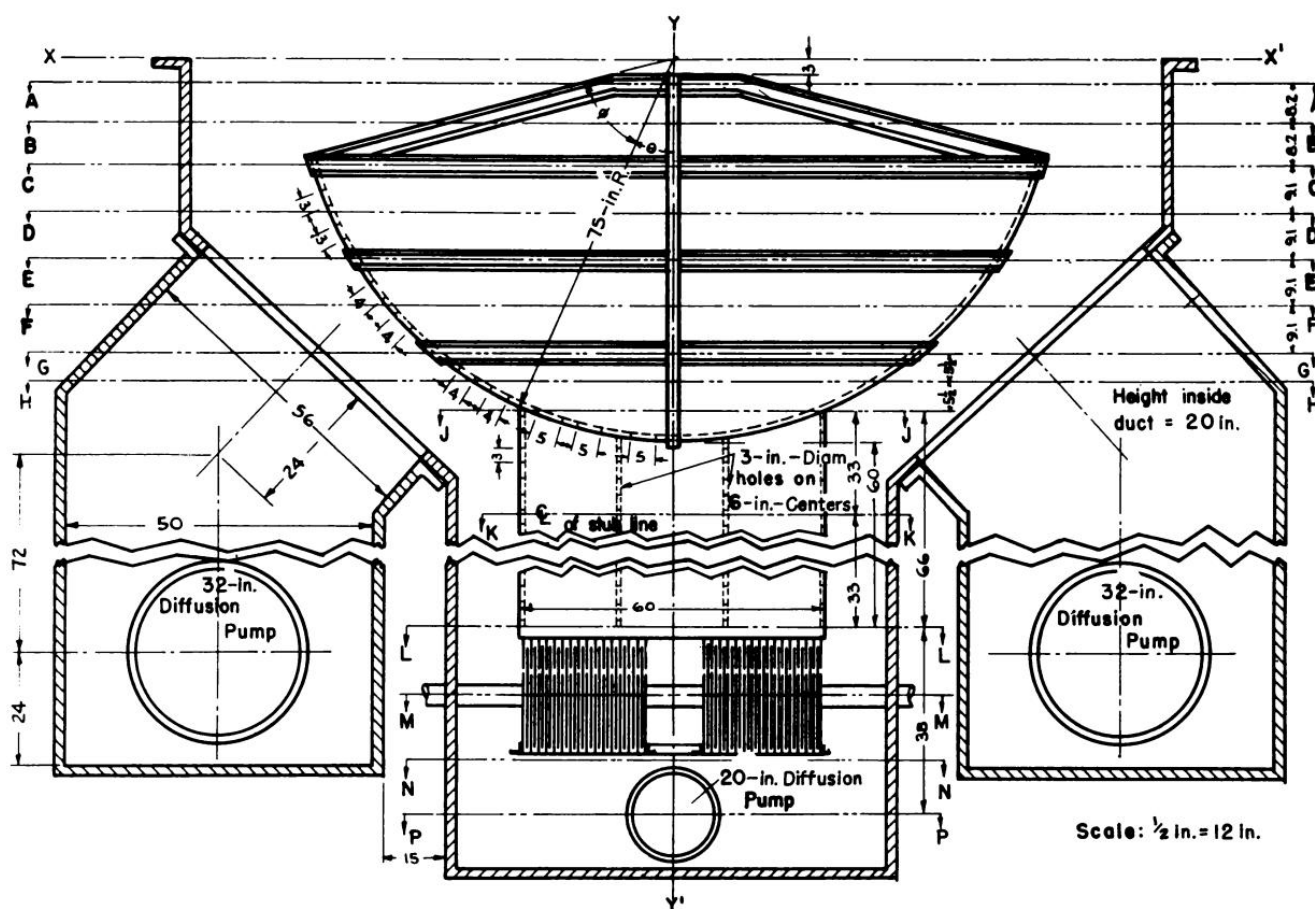


Fig. 4.2—Vacuum chamber in cross section. (All dimensions are given in inches.)

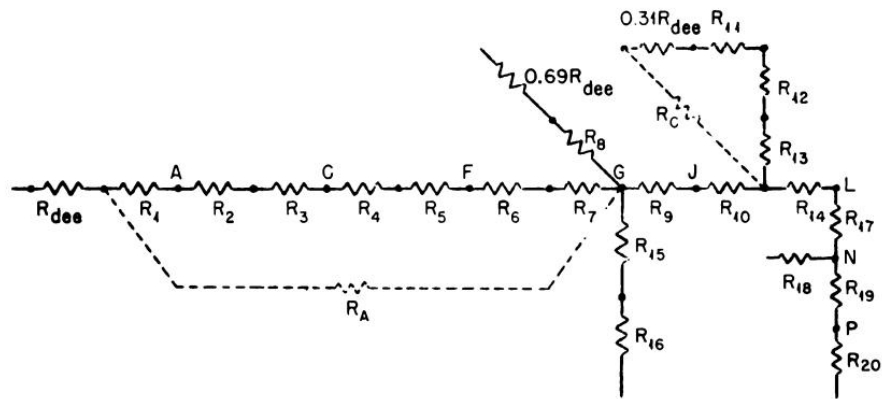


Fig. 4.3—Equivalent circuit of pumping "resistances" of vacuum system.

FIGURE 4.3 RESISTANCES DEFINED

(With reference to Fig. 4.2)

The "resistance" (reciprocal of the conductance) is shown for each element of the vacuum system (cf. Fig. 4.2) up to and including the diffusion pumps. In each case, except for R_{12} and the diffusion pumps, the conductance, corresponding to the resistances, as defined below, is computed on the basis of a rectangular duct with the cross-sectional area taken in a plane parallel to the X axis of Fig. 4.2. With reference to Fig. 4.2, the resistances are defined as follows:

R = total resistance inside dee up to section JJ

R_1 = resistance of orifice at section AA

R_2 = resistance of annular duct between sections AA and CC with average values taken at section BB

R_3 = resistance of orifice at section CC

R_4 = resistance of annular duct between CC and EE with average values taken at DD

R_5 = resistance of orifice at section EE

R_6 = resistance of annular duct between EE and GG with average values taken at FF

R_7 = resistance of orifice at section GG

R_8 = resistance of all orifices along back of dee between sections CC and JJ

R_9 = resistance of annular duct between sections GG and JJ

R_{10} = resistance of annular duct between sections JJ and KK

R_{11} = resistance of all the 5- by 2 $\frac{3}{4}$ -in. orifices along back of dee opening into tuning stub line

R_{12} = resistance inside tuning stub line taken with cross-sectional area along the Y axis of Fig. 4.2 and including resistance of 3-in.-diam orifices in reinforcing braces

R_{13} = resistance of the 3- by 2 $\frac{3}{4}$ -in. orifices along both sides of tuning stub line

R_{14} = resistance of annular duct between sections KK and LL

R_{15} = resistance of pump leads from sides of vacuum chamber up to the center line of 32-in. diffusion pumps

R_{16} = resistance of the two 32-in. diffusion pumps and their baffles

R_{17} = resistance of region surrounding rotating condenser and stator blades between sections LL and NN, i.e., the two outer ducts and the center duct, including the end corrections

R_{18} = resistance of annuli between stator and rotating condenser blades

R_{19} = resistance of duct between sections NN and PP

R_{20} = resistance of the 20-in. diffusion pump and its baffle.

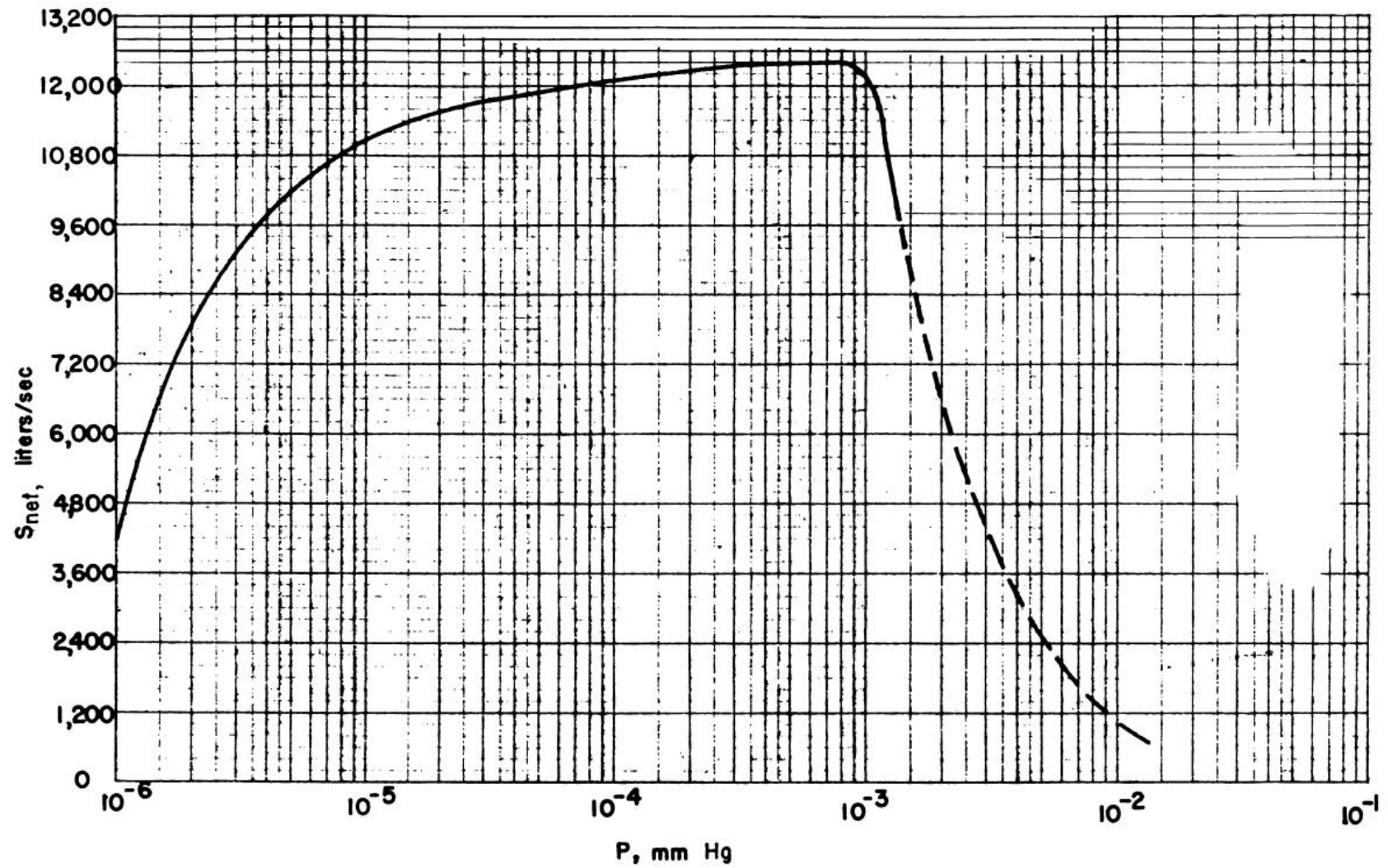


Fig. 4.4— S_{net} vs. pressure in vacuum chamber for two 32- and one 20-in. diffusion pumps.

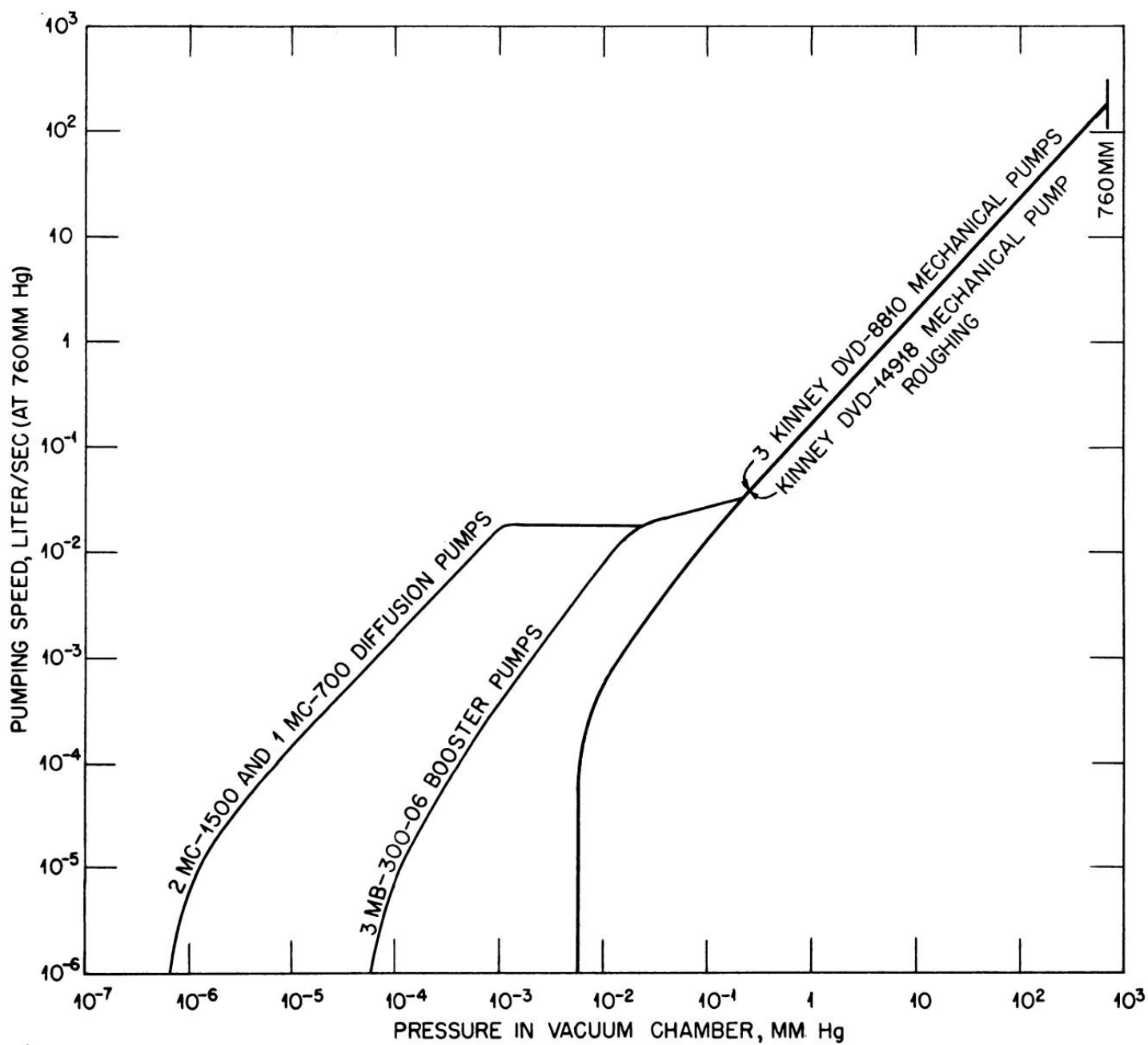


Fig. 4.5—Pump-speed curves for cyclotron vacuum system.

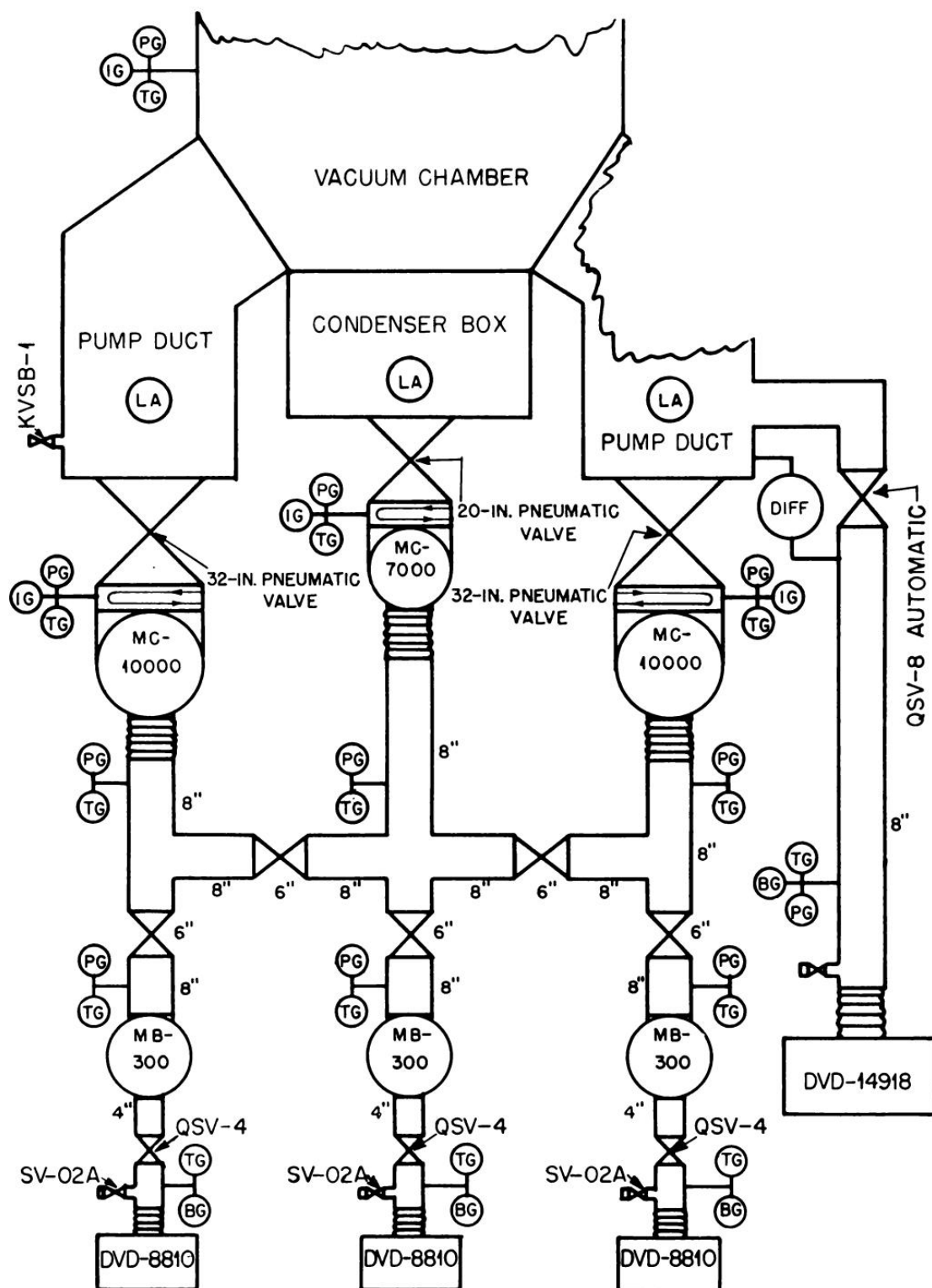


Fig. 4.6—Schematic of vacuum system. IG, ionization gauge; PG, Pirani gauge; TG, thermo-couple gauge; BG, Bourdon gauge; DIFF, differential gauge; , Freon baffle; , flexible coupling; LA, liquid-air trap. (All dimensions are given in inches.)

Technical Report No. 5

ROTATING CONDENSER FOR SYNCHRO-CYCLOTRON

By M. H. Foss

A design for a rotating condenser which has appropriate variation of capacity with time is developed. A temperature compensation method is discussed.

5.1 REQUIREMENTS FOR THE CONDENSER

1. The operation of the oscillator determines the variation of capacity with time which will keep the ion phase constant. This variation is not very critical. An approximate solution can be obtained by the following argument.

$$\frac{mV}{r} \sim Ef \sim He \quad (1)$$

where E = total energy H = magnetic field
V = ion velocity V_d = dee voltage
r = radius of orbit m = relativistic mass
f = frequency

The change in the total energy at constant phase is

$$\frac{dE}{dt} \sim f V_d \quad (2)$$

Using Eq. 1, its derivative, with respect to time (neglect dH/dt) and Eq. 2, gives

$$\frac{df}{dt} \sim -f^3 V_d \quad (3)$$

From the data on the three-fourths scale oscillator,* the relationship between f and V_d is known to about 10 per cent. From this data Eq. 3 was numerically integrated. From the resulting curve of frequency vs. time and the data giving frequency vs. capacity for the three-fourths scale oscillator, the curve of capacity vs. time was obtained. This is given in Fig. 5.1.

2. The required change in capacity was found from data on the three-fourths scale model to be $800 \mu\mu$, but the required minimum was not determined. This was estimated by Williams to be $70 \mu\mu$. It should be pointed out that a low minimum capacity increases the cost of the condenser.

3. The power supply is only able to drive the oscillator on a 50 per cent duty cycle so that the time for the return to the minimum useful capacity must be as great as the acceleration time. It is clear that more time used in the return is wasted and lowers the average beam current.

*See Technical Report No. 6.

4. The time required to accelerate the ions is 10^{-3} sec, when the phase is such that half the peak dee voltage is used on each acceleration (the condition for maximum average beam current).^{*} Since the average beam current is not a strong function of the time required to accelerate the ions, this time can be increased to some extent.

5. There is a minimum repetition rate at which the oscillator must be pulsed to give satisfactory operation of the power supply. This minimum is 100 per second but will be lowered by adding capacity in the power supply circuit.

5.2 DESIGN OF CONDENSER

These requirements are satisfied by a design with the general features taken from the design by Ankersen. Those features which are different are discussed here. The method of calculating capacity is discussed in Appendix I.

The required curve for capacity vs. time was obtained by laying out a shaped dee stator (see Fig. 5.2), which gave the capacity curve shown in Fig. 5.1, and by passing the best circle through the curved edge of the blades to facilitate construction (see Fig. 5.4); the estimated minimum capacity is not quite reached by this design, which uses eight blades at 45 deg on the rotor. It could be met with a seven-blade rotor but the greater complexity of machining this shape and the higher rpm required to obtain a given repetition rate seem to make the seven-blade design a poor choice.

The dee stator subtends 27 deg at the center of the rotor, but only about $22\frac{1}{2}$ deg is used to obtain the change of $800 \mu\mu$ required by the oscillator; thus the oscillator can be off 50 per cent of the time.

Since the $800 \mu\mu$ change in capacity is obtained in $\sim 22\frac{1}{2}$ deg of rotation, the ion acceleration time will be 10^{-3} sec if the shaft speed is 3750 rpm. The average beam current will be slightly reduced by making the acceleration time 1.25×10^{-3} sec and the rpm 3000, giving a repetition rate of 400 per second.

The teeth are spaced, as shown in Fig. 5.3, so as to minimize variation in capacity with change in temperature. The curve for capacity vs. temperature is shown in Fig. 5.5. The calculation is discussed in Appendix II.

The change in the maximum capacity due to systematic errors in construction may be estimated from the following table.

	Construction Error	% Capacity	Variation
1	Ground stator shifted, $\Delta = 0.002$ in., with respect to rotor (parallel to rotor center line)	0.11	Δ
2	Dee stator shifted, $\Delta = 0.004$ in., with respect to rotor (parallel to rotor center line)	0.10	Δ^2
3	Dee stator blades too thin, $\Delta = 0.0001$ in.	0.08	Δ
4	Ground stator blades too thin, $\Delta = 0.001$ in.	0.12	Δ
5	Rotor blades too thin, $\Delta = 0.0001$ in.	0.09	Δ

Those capacity changes which vary linearly with Δ tend to average out when the errors are random. Thus, if all the rotor blades are held to $+0.001$ in. thickness, an error of about 0.1 per cent might be expected in the capacity.

The capacity change due to shifting the dee stator blades varies with Δ^2 and will not average out.

^{*}D. Bohm and L. L. Foldy, Phys. Rev., 72: 649(1947).

APPENDIX I

CALCULATION OF CAPACITY

The capacity was calculated in four parts, (1) apparent capacity, from overlapping areas, (2) fringing capacity, from edge of plates covered by another plate, (3) stray capacity, from edge of plates not covered by another plate, and (4) end effects, capacity at the ends of the condenser bank.

Calculating the apparent capacity is straightforward. Calculating the fringing capacity was done by making flux maps of the interesting region and applying an appropriate correction, e.g., see Fig. 5.6. Calculating the strays was done by flux maps, e.g., see Fig. 5.7. When both fringing and stray capacity were present, the stray was mapped where the field strength associated with it exceeded the field strength due to the fringing flux (approximately). The end effects were estimated on the basis of a model condenser.

APPENDIX II

THERMAL EXPANSION OF ROTOR

The capacity between the uniformly spaced teeth of the condenser is

$$C_0 = kNA \left(\frac{1}{a} + \frac{1}{b} \right)$$

where k = constant

N = number of blades

A = area of blades

a = separation on one side of rotor blades

b = separation on the other side of the rotor blades

If the rotor expands a total amount (Δ) the capacity is

$$C = kA \sum_{n=1}^N \left(\frac{1}{a + \frac{n\Delta}{N}} + \frac{1}{b - \frac{n\Delta}{N}} \right)$$

Changing the sum to an integral and expanding in terms of Δ , this becomes

$$C = C_0 \left[1 + \frac{\Delta}{2} \left(\frac{1}{b} - \frac{1}{a} \right) + \frac{\Delta^2}{3} \left(\frac{1}{b^2} - \frac{1}{ab} + \frac{1}{a^2} \right) + \frac{\Delta^3}{4} \left(\frac{1}{b^3} - \frac{1}{b^2a} + \frac{1}{ba^2} - \frac{1}{a^3} \right) + \dots \right]$$

The formula for this condenser becomes (Δ in inches)

$$\frac{C}{C_0} = 1 - 0.308\Delta + 30.6 \Delta^2 - 55 \Delta^3 + \dots$$

This result is shown in Fig. 5.

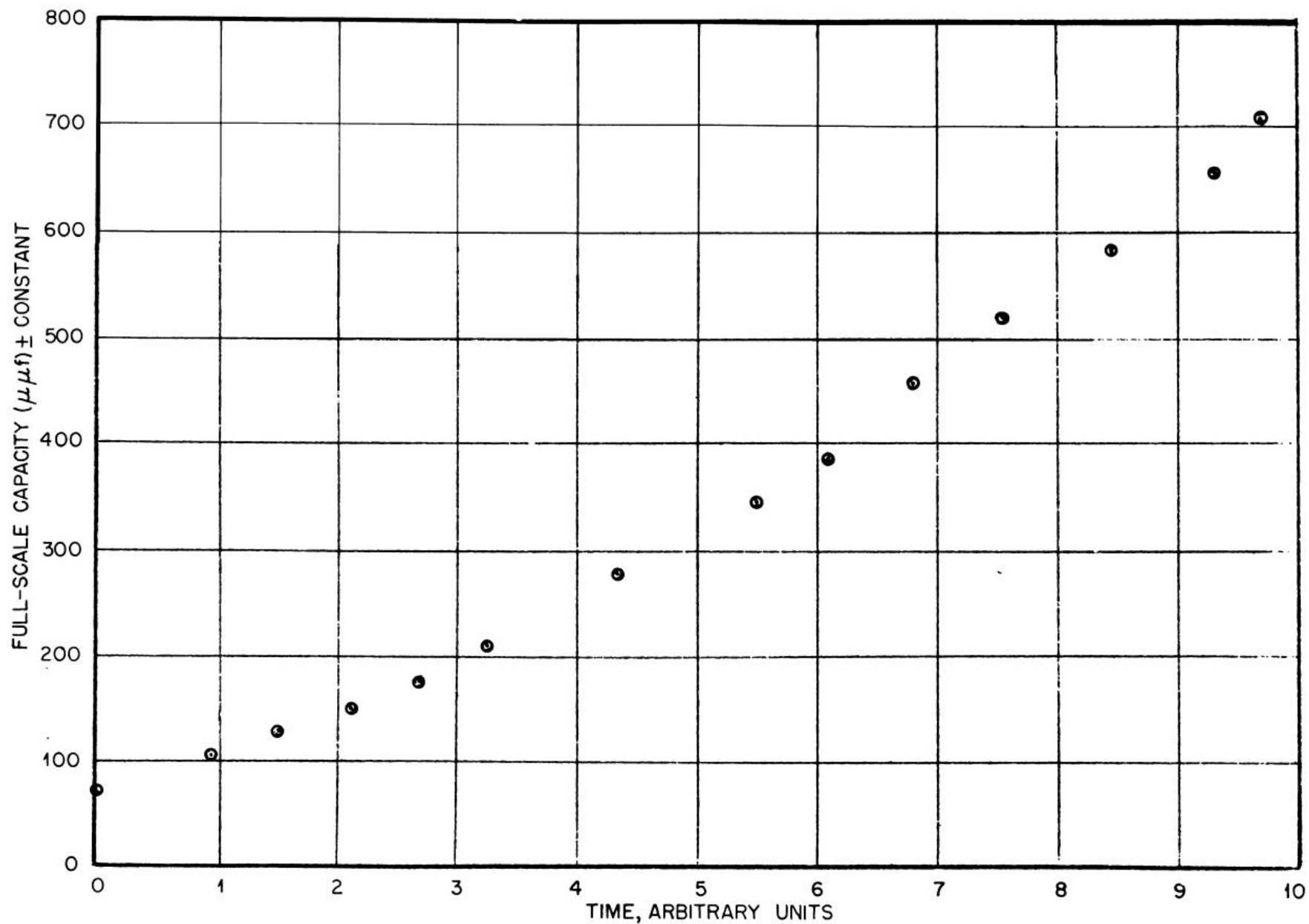


Fig. 5.1—Capacity vs. time obtained by integrating $\frac{df}{dt} \sim f^3 V_d$, using f vs. V_d and f vs. C from three-fourths scale model oscillator. f , frequency; V_d , dee voltage; C , capacity.

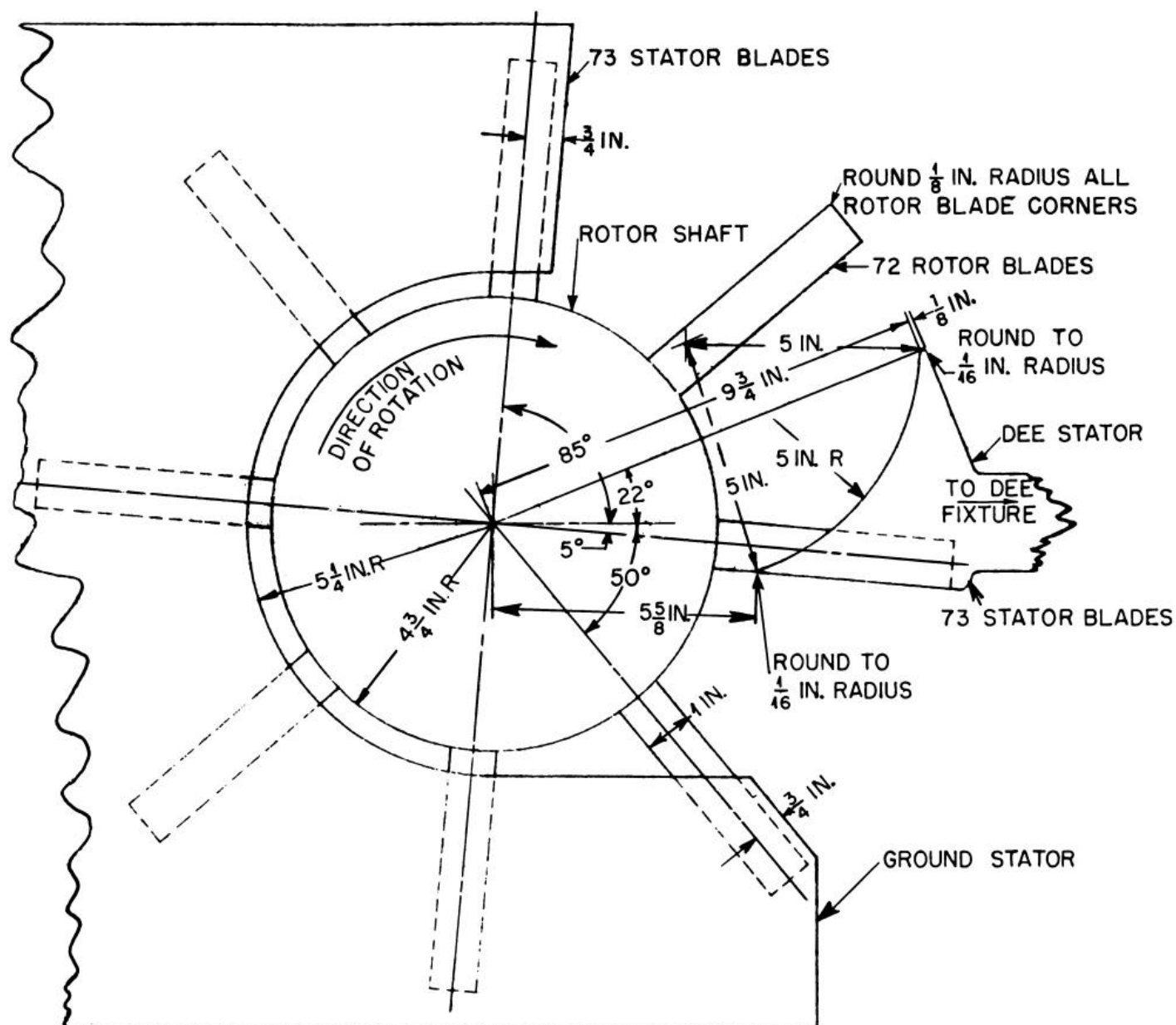


Fig. 5.2—Condenser in position of maximum capacity. Scale, 1 in. = 3 in.

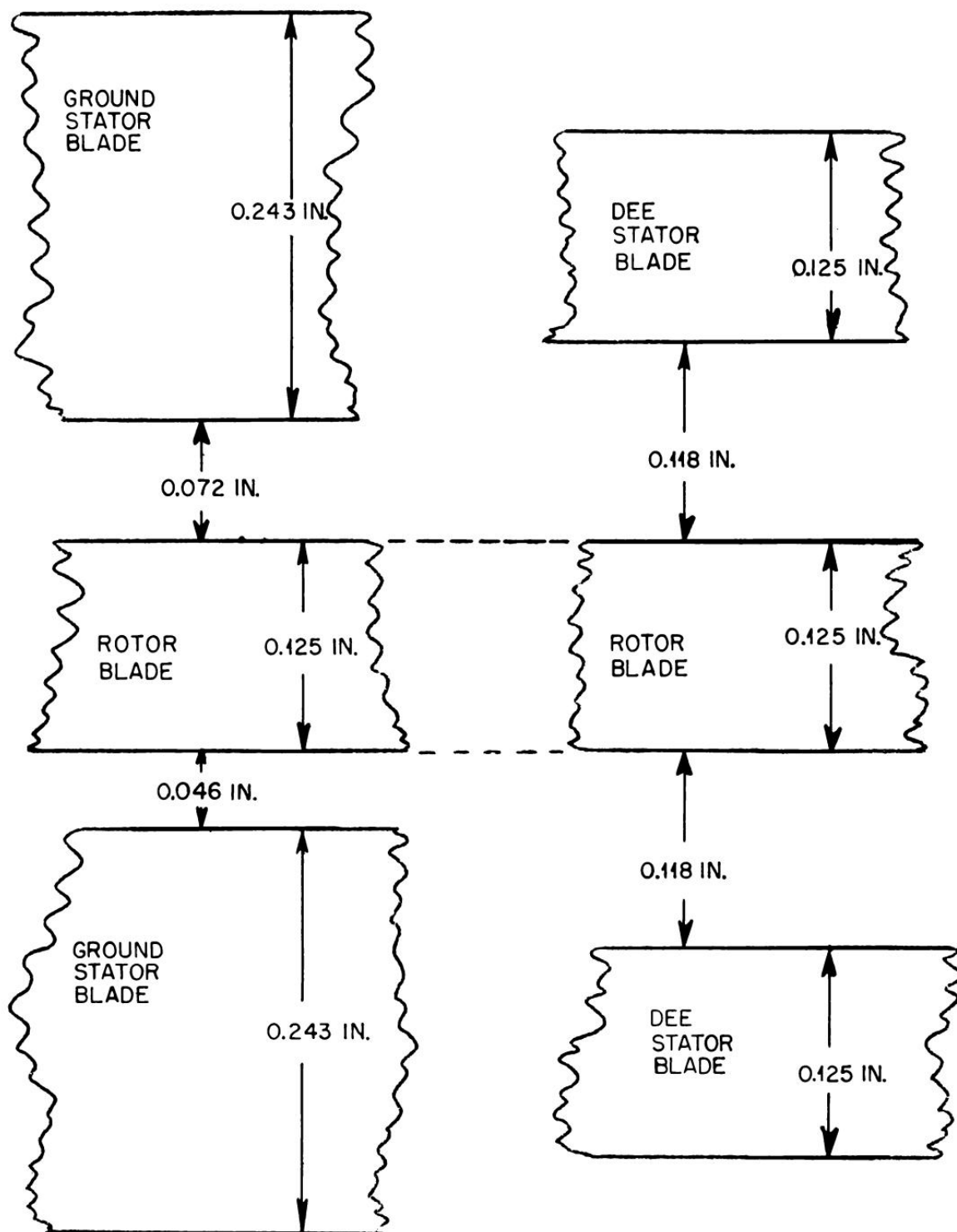


Fig. 5.3—Axial positions of condenser blades. Scale, 10 = 1. (The bearing below the bottom of the paper is fixed to the shaft; the shaft slides through the bearing above to allow for thermal expansion of the shaft.)

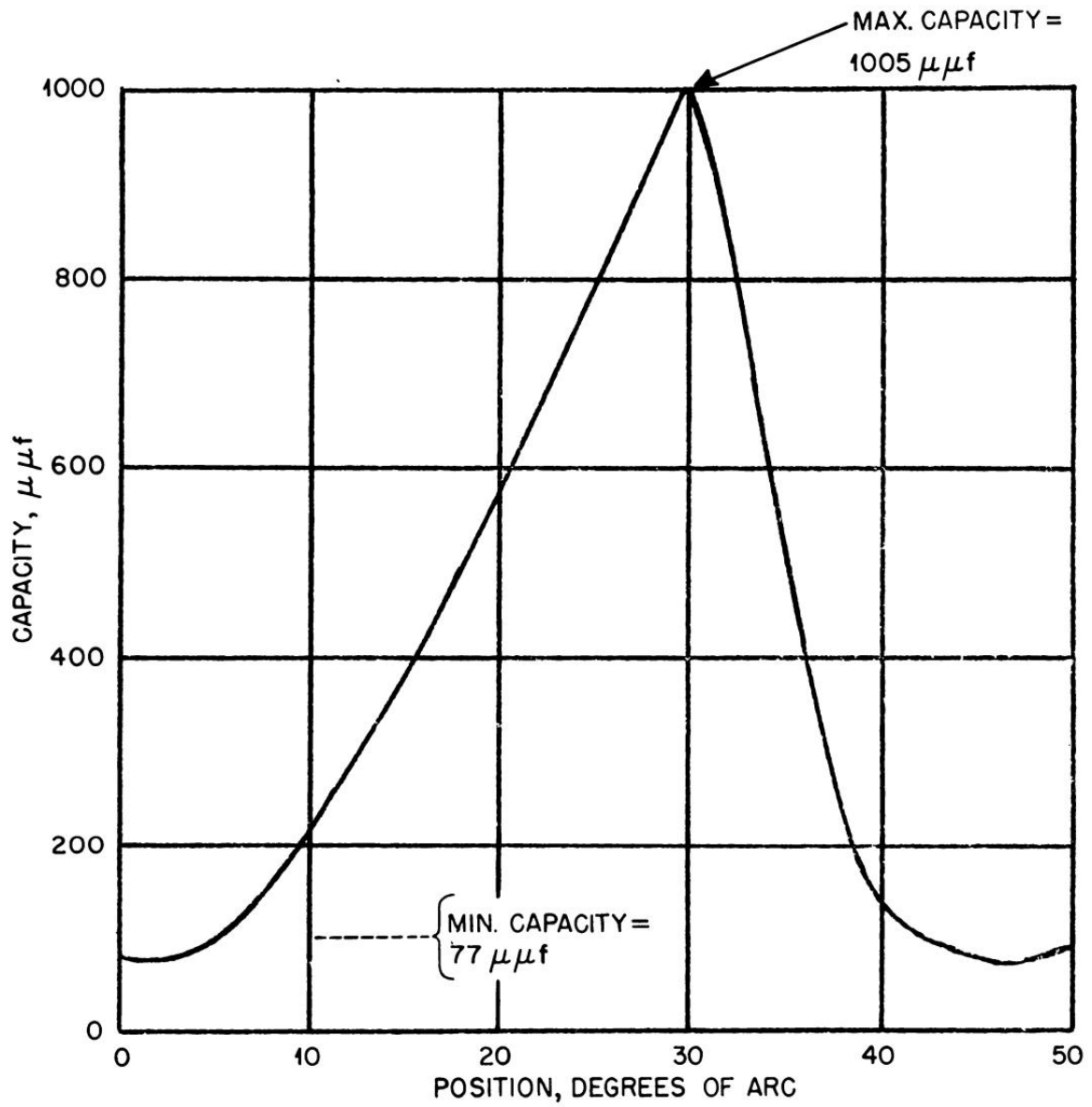


Fig. 5.4—Capacity vs. position of rotor.

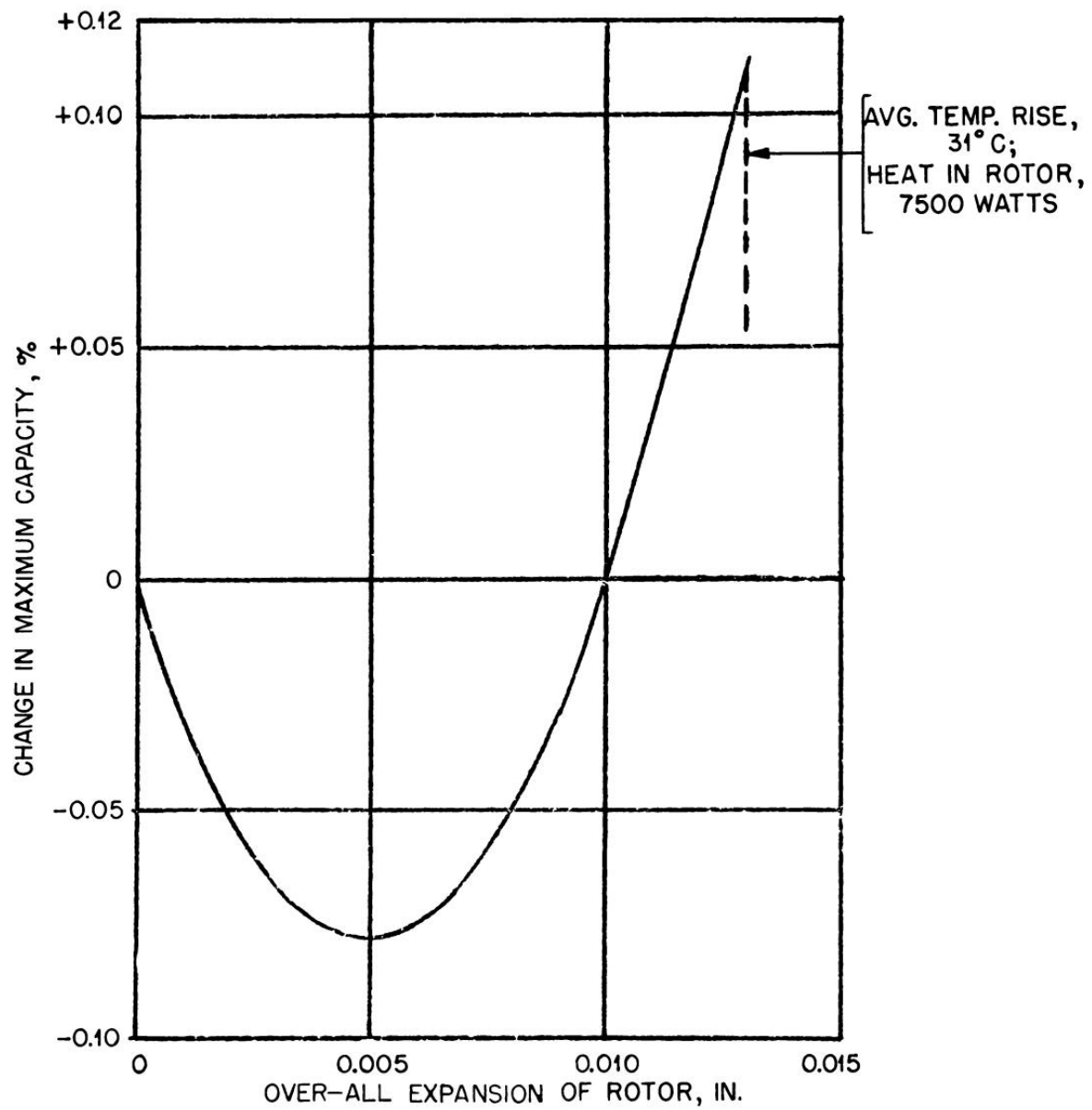


Fig. 5.5—Maximum capacity vs. expansion of rotor

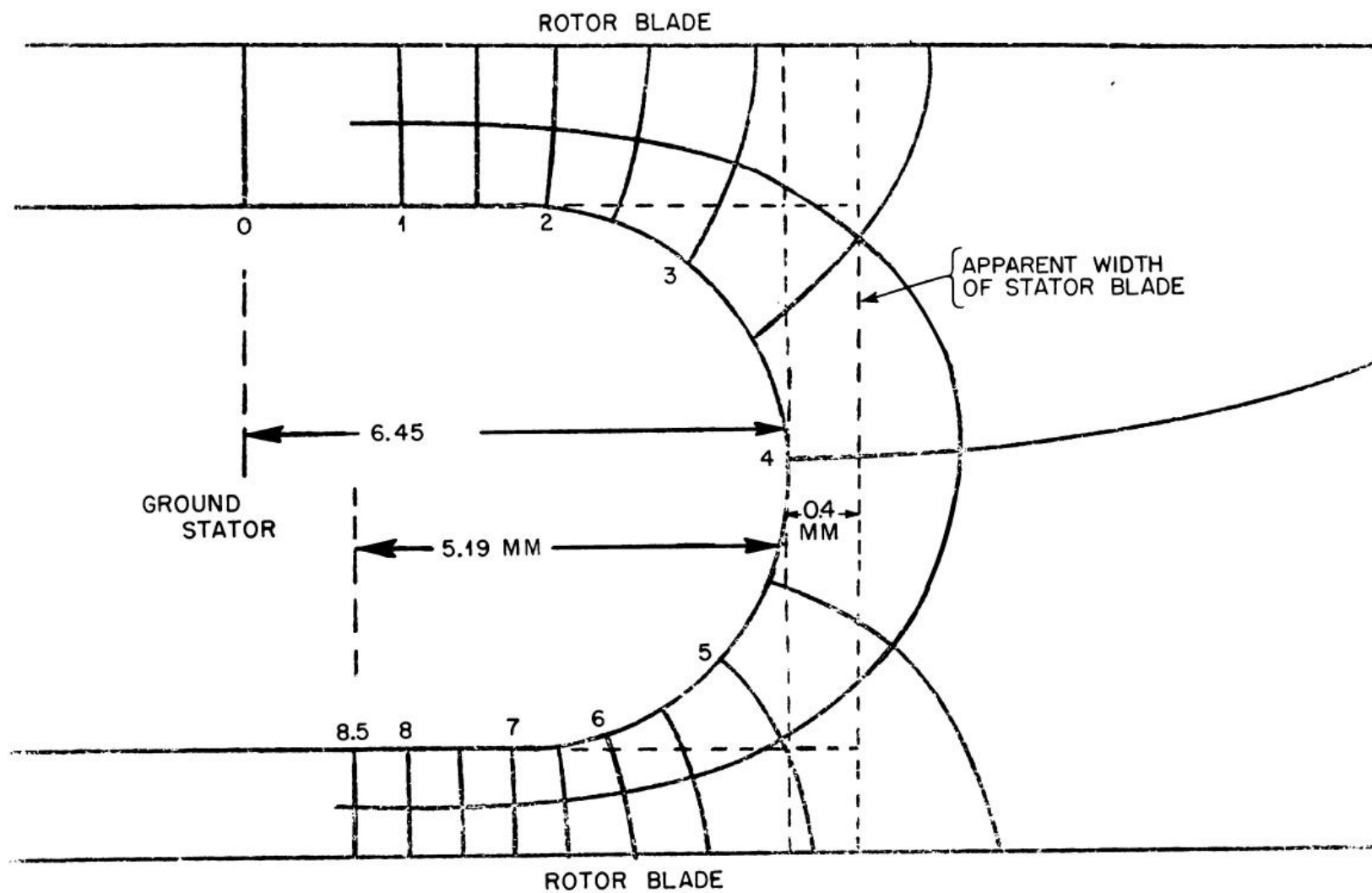


Fig. 5.6—Typical "fringing flux" flux map.

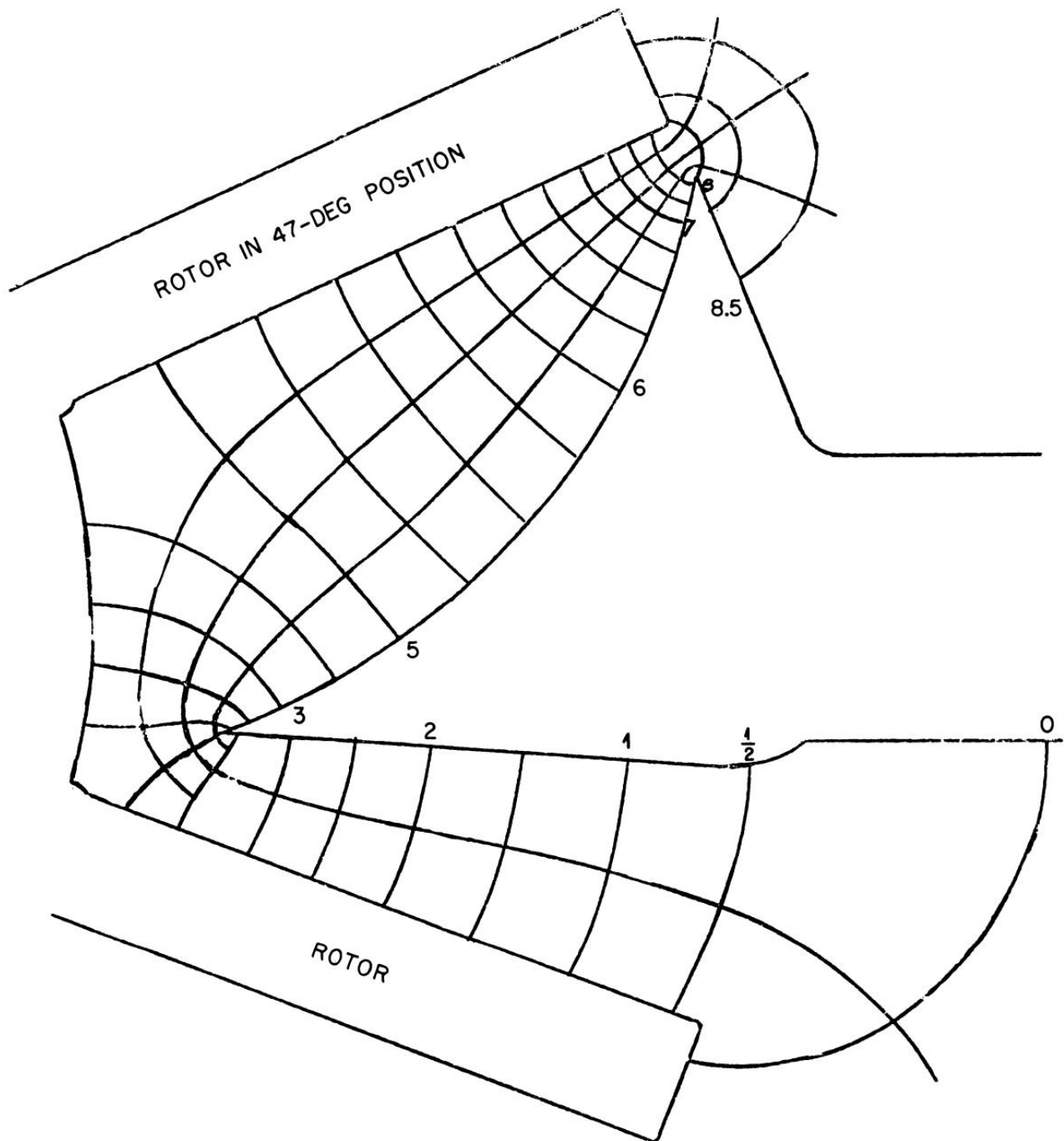


Fig. 5.7—Stray capacity in position of minimum capacity.

Technical Report No. 6

THREE-FOURTHS SCALE MODEL OSCILLATOR REPORT

By E. M. Williams, L. W. Johnson, and H. M. Collins

6.1 CONSTRUCTION

The essential features of construction of the final version of the three-fourths scale model oscillator for the Carnegie Institute of Technology synchro-cyclotron are shown in Figs. 6.1a, b, and c, and in the photograph of Fig. 6.2. The principal components are the dee, the extension of the back of the dee known as the tuning line, and the three stubs. The two outer stubs serve as part of the frequency-determining system, whereas the center stub also serves this purpose and acts as the coupling system for the oscillator tubes. The oscillator stub is also equipped with a small coupling loop, shown in Fig. 6.1c, used to load the unwanted mode, which on the three-fourths scale model was about 50 megacycles, in which the oscillator stub forms a capacity-loaded half-wavelength line.

Tuning capacity in this model is adjusted by a swinging grounded electrode opposite the exposed end of the tuning line. This plate is visible in Fig. 6.2, together with the screw drive by which the tuning capacity is adjusted.

In most of the tests with this model the oscillator grids were connected by means of straps, shown in Fig. 6.1b, to adjacent points on the tuning stubs. This connection was necessary to ensure suppression of the 50-megacycle mode.

6.2 TEST RESULTS

Results of the tests yielding the highest dee voltage are shown in Fig. 6.3 for a frequency range corresponding to 19 to 32 megacycles in the full-scale model, except that the limits are 5 per cent higher than they should be for the scale factor used. For pulsed operation with a duty cycle of 50 per cent, the test results indicate an input of 75 kw at 12.5 kv, which is practicable with the power supply available. The fairly low efficiency, 27 per cent, indicates that it would be desirable to go to six type-880 tubes in the final model, especially since power-supply capacity is available for the additional tubes.

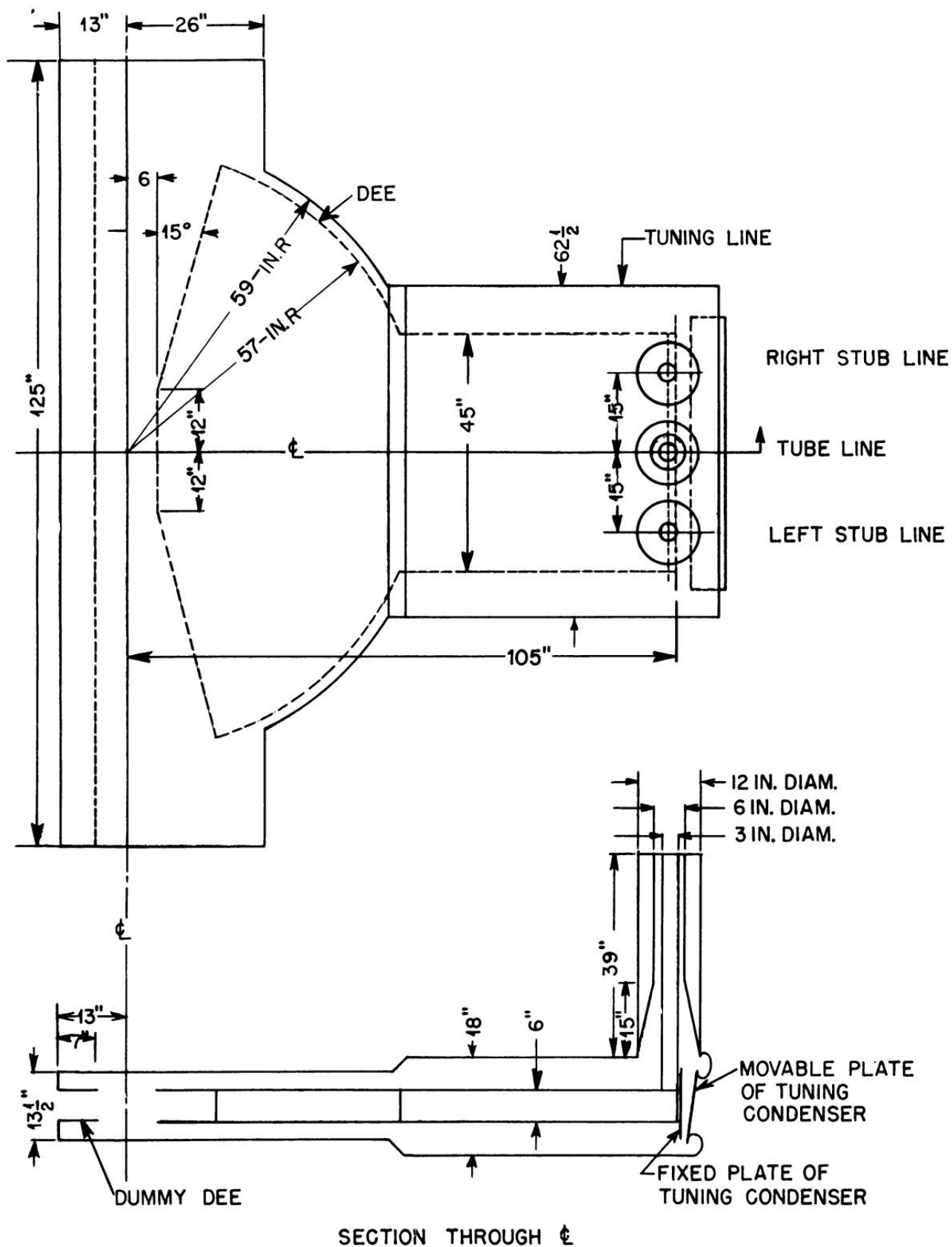


Fig. 6.1a—Three-fourths scale model oscillator assembly, showing dee, tuning line, stubs, and tuning capacity. (All dimensions are given in inches.)

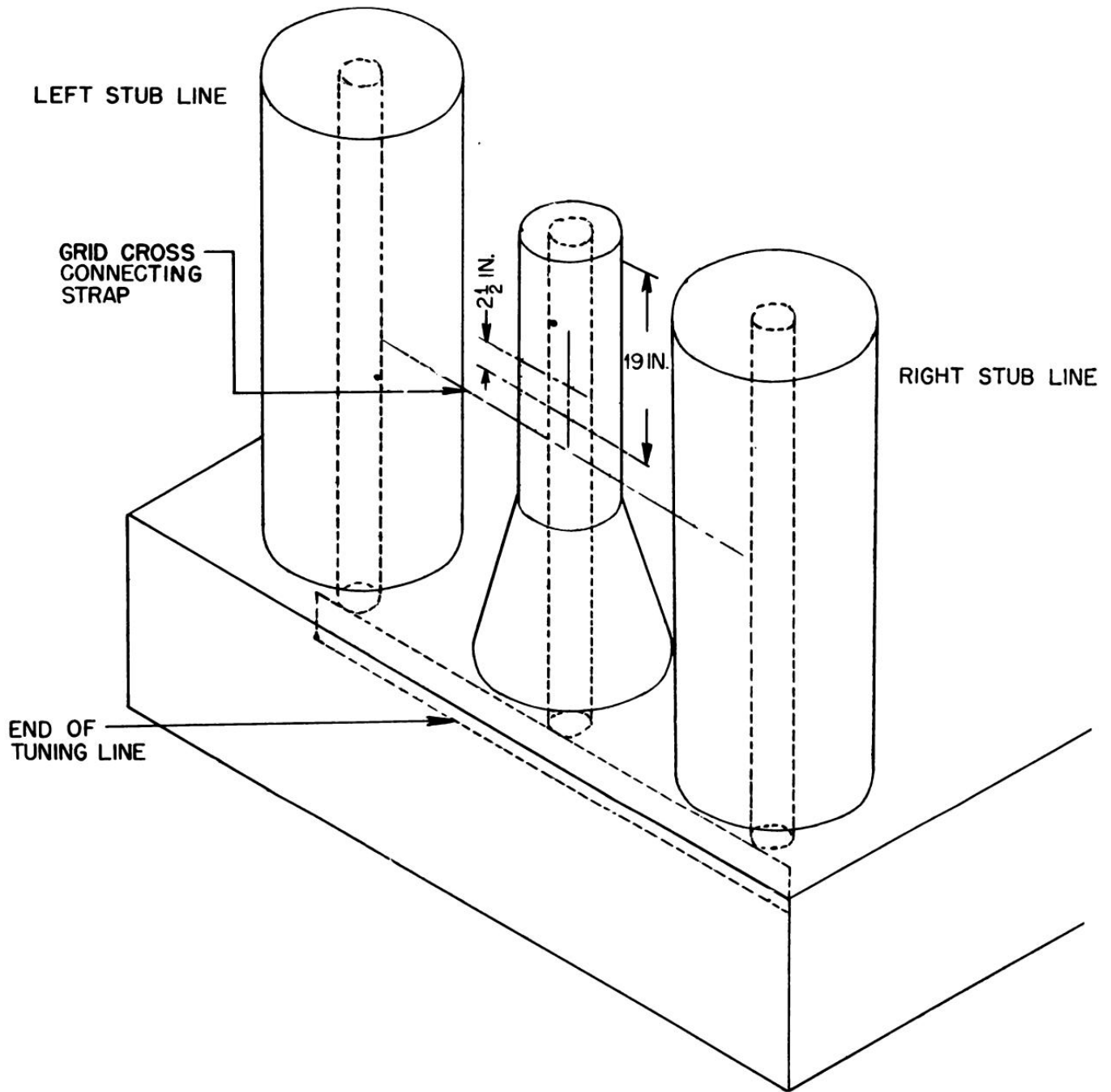


Fig. 6.1b—Stubs and end of tuning line with oscillator tubes and tuning condenser removed.

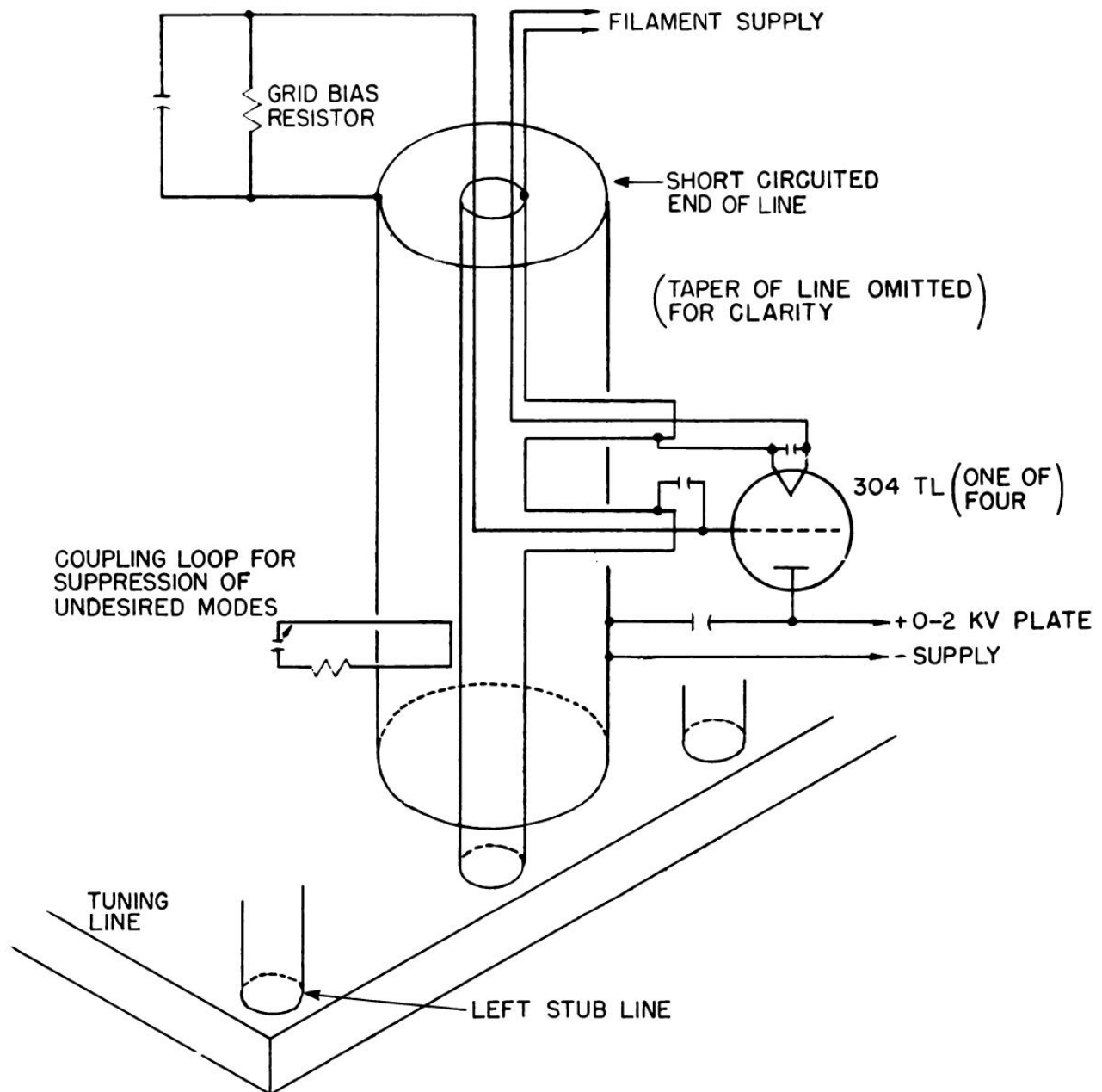


Fig. 6.1c—Semischematic diagram of center (tube) stub line. Only one of four tubes is shown; connections for other tubes are made in an identical manner.

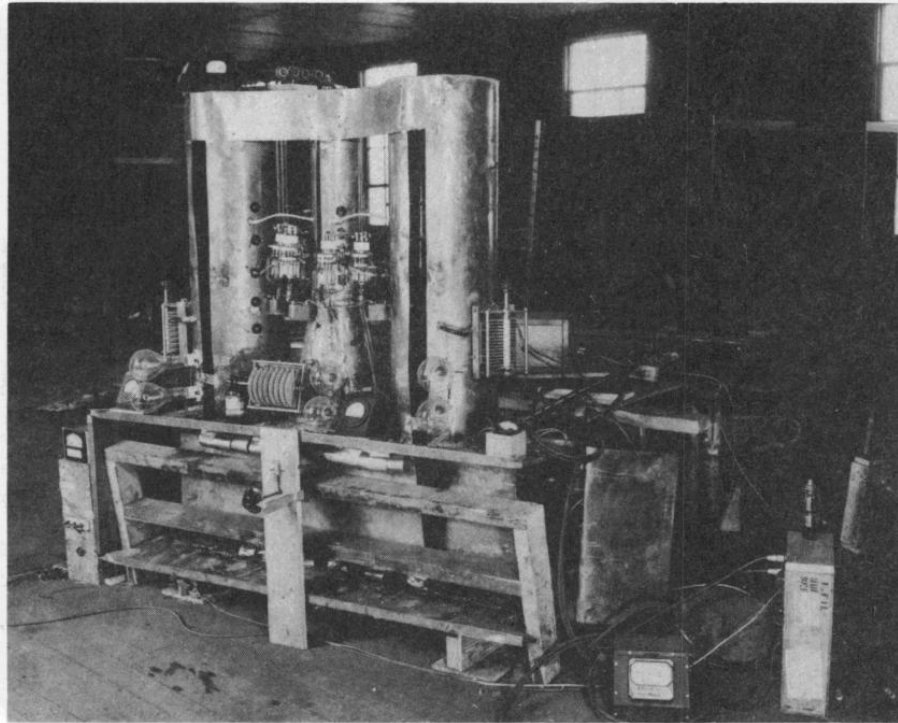


Fig. 6.2—Photograph of three-fourths scale model oscillator.

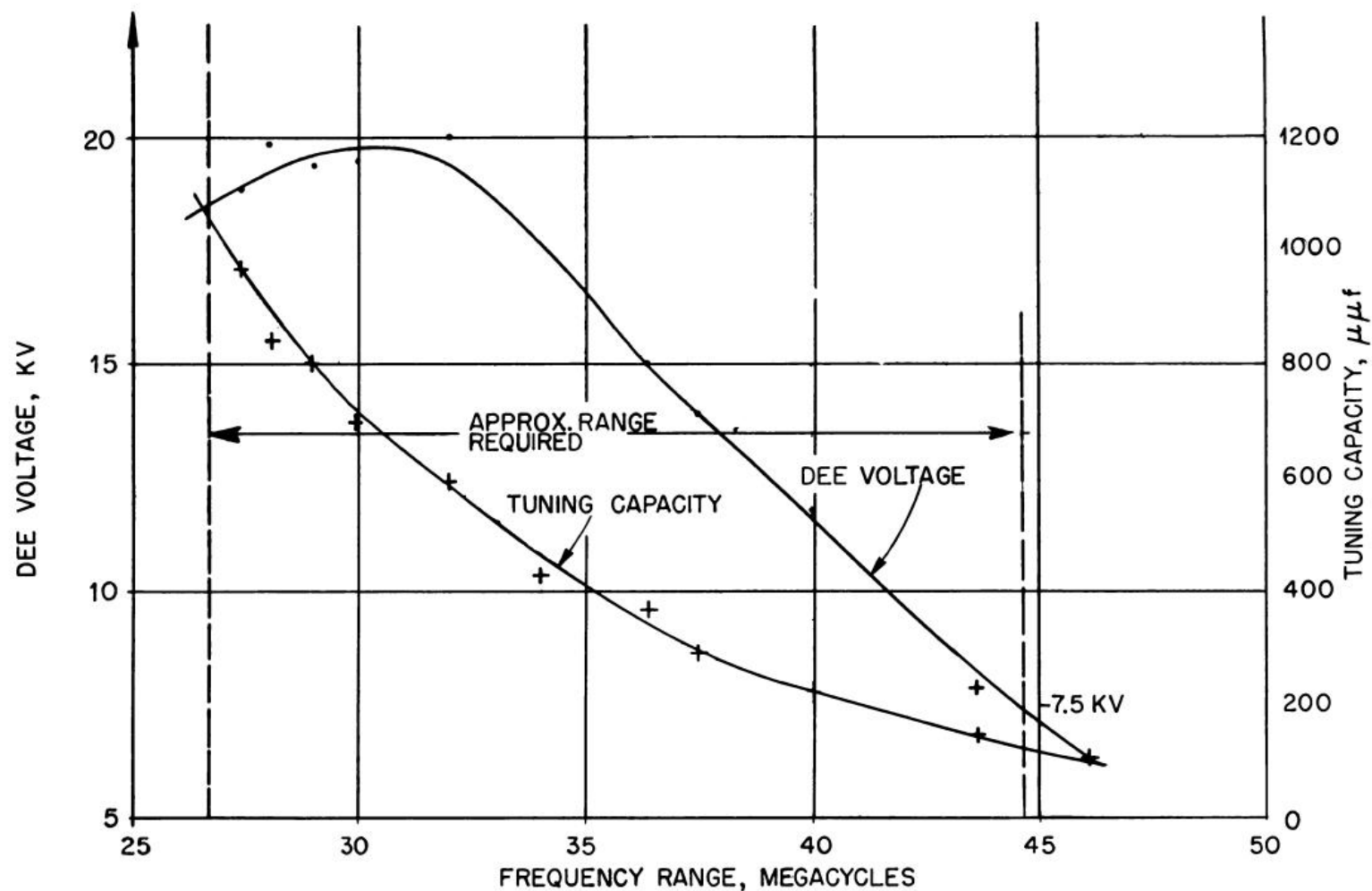


Fig. 6.3—Typical characteristics of three-fourths scale model, adjusted to correspond to operation of four type-880 tubes operated at 12.5-kv anode voltage. Average conditions for 12.5-kv anode voltage, 150-kw input, 27.5-kw plate dissipation per tube for approximately linear frequency variation.

Technical Report No. 7

DISCRIMINATOR DESIGN AND CONSTRUCTION FOR SYNCHRO-CYCLOTRON

By C. H. Grace

7.1 NECESSITY FOR DISCRIMINATOR

The synchro-cyclotron accelerates ions in a circle of gradually increasing radius. When the particles achieve a radius at which the field starts to become nonuniform, the group of ions is deflected out of the magnet air gap. The discriminator described in this report determines the time of operation of the deflector.

The group of ions to be deflected is distributed along its orbit. The deflector-plate voltage must be applied when the group is in a certain position in its orbit. The design problem was to operate the deflector at the proper instant.

The orbital radius is a function of the dee-oscillator frequency. A measure of frequency is therefore a measure of particle radius. However, it is impossible to design equipment which measures the frequency so accurately that even the phase of the dee-oscillator electric wave is determined. The equipment thus divides into a frequency-sensitive circuit and a phase-sensitive circuit. The frequency-sensitive circuit "gates" the phase-sensitive circuit, and the phase-sensitive circuit triggers the deflector.

Perhaps the more difficult of these problems is the frequency-sensitive circuit. That is the equipment described in this report.

This circuit must determine the instantaneous frequency of a wave whose frequency is changing at a typical rate of 2500 megacycles/sec².

The dee-oscillator frequency at which the deflector operates must be manually variable. The operator may deflect the beam at whatever particle energy (orbital radius) he desires, within limits. This control is smoothly variable rather than step-variable. The feature that the deflection time be smoothly variable necessitated somewhat more complex circuits than would have been otherwise required. (See Fig. 7.1.)

7.2 SIMPLIFIED DESCRIPTION OF OPERATION

The dee oscillator produces an f-m wave as shown in Fig. 7.2. It may be desired to produce a pulse for the deflector when the dee-oscillator frequency reaches point P. The discriminator could do that by adjusting a band-pass filter for the frequency P. Then when the f-m wave sweeps through the filter pass band, an a-c transient would be passed by the filter.

An inductance-capacity filter tends to change frequency slightly. An LC filter of the degree of frequency stability required cannot be built. A crystal filter could be made sufficiently stable (if the crystals were temperature controlled). However, they do not permit the deflection frequency to be smoothly variable.

The simplified block diagram of Fig. 7.3 shows the method used. Block 8 is a band-pass filter of about 1-megacycle center frequency. Block 4 is a local oscillator with a frequency about 1 megacycle below the desired deflection frequency P. When the dee-oscillator signal sweeps through point P, the 1-megacycle filter passes an a-c transient. The shape of the transient is shown in Fig. 7.3. The transient is rectified and its envelope triggers a pulse generator.

Accuracy is obtained by temperature-controlling the local oscillator and its crystals. The components of the band-pass filter are also enclosed in an oven.

Deflection frequency is made variable by making the frequency of the band-pass filter variable. The frequency of the band-pass filter may be adjusted by the operator to any value between 1 and 1.25 megacycles. (The total range of deflection frequency was extended, by switching of local oscillator crystals, to a span of $2\frac{1}{2}$ megacycles.)

7.3 DETAILED DESCRIPTION OF OPERATION AND CALIBRATION

The preceding description of the discriminator is greatly simplified. Figure 7.4 gives a more complete diagram, and the following paragraphs describe the complete discriminator equipment.

A sample of the dee-oscillator output is transmitted to the discriminator by means of a 50-ohm coaxial line. The signal is mixed with the "operate" oscillator, Block 4, operate-oscillator frequency being about 1 megacycle lower than that of the dee oscillator. The mixer, Block 7, contains in its output circuit a current component at the "difference frequency" between the dee oscillator and the operate oscillator. The operate-oscillator frequency is constant (and accurately known). The frequency of the dee oscillator varies with time. A graph of the frequency of the dee oscillator vs. time is shown in Fig. 7.2. The output of the mixer, Block 7, varies in frequency in a manner similar to the dee oscillator, but the mixer output is at a frequency near zero. In fact, it sweeps through zero. The plate circuit of the mixer is a tuned circuit, Block 8, which transmits only those frequencies in its pass band. Thus the output of filter No. 8 is zero, except when the difference between dee-oscillator frequency and operate-oscillator frequency lies near the center frequency of the filter. When the difference frequency sweeps through the pass band of the mixer's plate circuit (filter), an a-c transient is passed. That a-c transient is shown in Fig. 4. (Notice that the length of the envelope of this wave between half power points is of the order of $16\ \mu\text{sec.}$) This transient is rectified (No. 10) then used to trigger a "one-shot" multivibrator, Block 15. The multivibrator's square-wave output is differentiated, yielding a narrow positive pulse at the beginning of the square wave (leading edge), and a narrow negative pulse at the trailing edge of the square wave. (The negative pulse is suppressed since it is not needed.) The positive pulse is the required output of the discriminator. Its leading edge is produced at the exact instant at which the frequency of the dee oscillator reaches the desired deflection frequency.

Calibration of Filter No. 8

The simplified block diagram of Fig. 7.5 shows how the band-pass filter is calibrated. This filter is smoothly variable from 1 to 1.25 megacycles and is calibrated only at its end-points. The "calibrate" oscillator is 18 megacycles.

The operative oscillator, Block 4, is first set at 19 megacycles, and the variable-frequency filter, Block 8 at 1 megacycle. The mixer (No. 7) output is then 1 megacycle, and the filter may be trimmed for maximum response to this signal. (Trimming is accomplished with a calibrating control.) Response of filter No. 8 is observed on a meter. In order to increase the accuracy with which the exact resonant point may be located, the subtractor, No. 21, is used to deduct the lower portion of the response. Thus a more sensitive meter may be used. Also, the sharpness of resonance of No. 8 is increased during calibration by means of switching. The resonant frequency is not changed by this. The subtractor, No. 21, was required in the equipment for automatic pulse-height control anyway, so it was made to serve two purposes.

To calibrate the filter at 1.25 megacycles, the operate oscillator selects the 19.25-mc crystal, whereas the calibrate oscillator remains at 18 megacycles.

The accuracy of filter No. 8 is greatly improved by enclosing its components in the oven, as was pointed out by Foss. Thus errors due to change in resonant frequency caused by temperature changes are reduced.

Switching of Inputs to Mixer

When the filter No. 8 is being calibrated, it is necessary to switch one input of the mixer from the dee-oscillator sample signal (No. 2) to the calibrate oscillator (No. 3). Since switching at these frequencies is often troublesome, "electronic switching" of the two signals is employed. The screen supply voltage for the dee-oscillator preamplifier, No. 2, is disconnected when calibration is to be performed. By the same switching operation, screen voltage is supplied to the calibrate oscillator buffer, No. 3b. The screen of this tube was previously dangling. By allowing the screen of the unused pentode (whichever it may be) to dangle, the transconductance of the tube is made even lower than it would be if these screens were connected to ground when it was desired to turn off that tube. When the screens dangle, they quickly become negative by collection of screen current. When the dee-oscillator preamplifier, No. 2, is on, the calibrate oscillator buffer, No. 3b, is off, and vice versa.

Oscillators, Blocks 3 and 4

The frequency of the operate oscillator, No. 4, may change slightly. Any change in that frequency results in a firing inaccuracy exactly equal to the frequency error of that oscillator.

In order to minimize the frequency error, the entire oscillator was enclosed in an oven. The oven is thermostatically controlled so that the oscillator crystals operate at 140°F. These crystals are of a type which theoretically have a zero temperature coefficient of frequency, but production inaccuracies prevent it. Aside from production tolerances, a zero coefficient is possible at only one temperature. At temperatures above 140°F, the temperature coefficient is negative, i.e., the frequency decreases with temperature increase; below the frequency of zero coefficient, the crystals have a positive coefficient.

The crystal oscillators, Nos. 3 and 4, will maintain a constant frequency within one part in 10,000. Since the resonant frequency is in the range of 20 megacycles, an error of ± 2000 cycles is the greatest expected error. These are overtone crystals with a fundamental frequency less than 1 megacycle. They operate on a strong high harmonic.

In order to decrease the loading on the tank circuit of the oscillators, buffer amplifiers were used, these buffers being operated at a bias where grid current is negligible (Blocks 3b and 5). Furthermore, grid resistors are omitted in the buffer amplifiers to increase Q. Transformer coupling to the tank circuit is employed.

Buffer Amplifiers Nos. 5 and 6

The plate circuit of the operate-oscillator buffer is a broad-banding circuit, No. 6. It is a "dead end" low-pass filter. One of the shunt arms of the filter is the input capacitance of the following stage, No. 7. Another of the shunt arms is the output capacitance of the operate-oscillator buffer itself, No. 5. The output signal travels down this two-section filter to a dead-end resistance termination. The output to the next stage is taken off between the two cascade pi sections of the filter. This arrangement gives the theoretical maximum gain-band width product achievable for one tube. Distributed amplification can produce greater gains per stage than this arrangement, but each stage of a distributed amplifier has several tubes.

Smoothing Filter, Block 14

The a-c transient produced by the mixer is about 16μ sec long. Since the frequency of the carrier of the a-c transient at the output of No. 8 is about 1 megacycle, there are about 16 cycles included between half-power points of the envelope. After this transient is rectified (by No. 10), it consists of unidirectional pulses occurring about 1μ sec apart, and this rectified signal must be used to trigger the multivibrator very accurately. If the wave form were not filtered to remove the 1-megacycle component, errors in multivibrator firing would be in multiples of 1μ sec. Such errors would probably exist since the phase of the 1-megacycle signal does not bear a constant relationship to its envelope. The 1-megacycle signal enters the pass band of the band-pass filter, No. 8, with a random phase. In order to eliminate the potential error source in multivibrator firing time which was just described, the rectified transient at the output of No. 10 must be filtered. Unfortunately it is theoretically impossible

to filter a transient without introducing time delay. The time delay thus introduced was kept to a minimum by using the least possible amount of filtering in No. 14, which would smooth the wave form. The time delay still remaining was compensated for by adjusting the bias of No. 19 of the multivibrator. The multivibrator, No. 15, was adjusted to fire at a smaller voltage, therefore at an earlier point on the wave form.

Automatic Pulse Height, Blocks 21, 22, 26, 27, 23, 24, and 16

The signal sample of the dee oscillator may change slightly in amplitude. These changes could result from "holes" in the frequency vs. amplitude characteristic of the dee oscillator. Thus, when the discriminator is set to produce a pulse at a particular frequency, the sample signal taken from the dee oscillator may be of a different amplitude than the sample signal received by the discriminator when it is adjusted to produce a pulse at some other frequency. That would have serious consequences, since the size of the pulse at the multivibrator grid, No. 15, would be unknown. The multivibrator would be caused to fire early by an extra-large-amplitude pulse on its grid. In order to eliminate this error source, an automatic pulse-height circuit was devised. This circuit could not be patterned after a conventional automatic volume control because the average current represented by the narrow triggering pulses is extremely small. Thus the automatic pulse-height circuit had to work on a peak-reading principle. The peak-reading principle was desirable in this case for another reason, namely, because the peak of the pulse at the output of filter No. 10 is used to calibrate the multivibrator firing voltage, and therefore is a primary quantity which should itself be regulated. The peak-reading rectifier, Block 23, had to be driven by a very-low-impedance generator in order that the time constant, which is the product of the generator's internal impedance times the capacitance which makes the circuit peak reading, might be comparable to the length of pulse measured. The output of this peak-reading rectifier, No. 23, automatically adjusts the gain of one of the pulse amplifiers. In order to maintain the pulse height at multivibrator grid at exactly the desired voltage, a subtracter was used. The subtracter circuit, No. 21, subtracts exactly 15 volts from the pulse height at the output of No. 10. The amount by which the pulse height exceeds 15 volts is used to operate the automatic pulse-height bias on No. 16. The automatic pulse-height circuit has a high gain. It is extremely effective in maintaining a constant height of pulse at the multivibrator grid. Note that no clipper action is employed. The shape of the pulse at the multivibrator grid is always the same (for any one repetition rate) regardless of input amplitude. Insertion of the detector element of this control, No. 21, at the multivibrator grid enables it to compensate for changes in signal amplitude regardless of whether they originate in the dee oscillator or the discriminator amplifiers.

Manual Gain Controls, Block No. 12

A manual gain control is provided on one of the pulse amplifiers. This is to adjust the feed-back gain (the gain of the complete automatic pulse-height feed-back loop) to a desirable operating point. This gain control consists of a variable cathode resistor used on a remote cut-off television pentode. (It is interesting to note that when the gain of this stage, No. 12, is changed manually, the gain of the preceding stage, No. 16, automatically changes in a direction to compensate. Thus the gain of these two stages, Nos. 16 and 12 together, as a cascade amplifier, is independent of the setting of the manual volume control. Thus the setting of the manual volume control adjusts the bias on the preceding stage, No. 16, indirectly. That is, it changes the feed-back voltage of the automatic pulse-height circuit so that the best operating bias for the automatic pulse-height amplifier, No. 16, may be realized.)

Block 19, Compensation for Changes in Repetition Rate of Dee Oscillator

The length of the transient at the output of filter No. 8 is a function of the repetition rate of the dee oscillator. As a result of this, the pulse which triggers the multivibrator, No. 15, does not always have the same shape or duration. The peak of the a-c transient does not occur at the instant at which the dee-oscillator frequency equals the resonant frequency of the tuned circuit, No. 8. Instead, the time at which the a-c transient is at a peak is somewhat later than the time at which the dee-oscillator frequency is in the center of the filter pass band. The amount that the peak is delayed is a function of the band width of

filter No. 8 (i.e., the steady-state frequency response curve of No. 8), as well as a function of the rate at which the frequency of the dee oscillator is changing. The repetition rate of the dee oscillator therefore affects the shape of the a-c transient and thus the shape of the pulse which triggers the multivibrator, No. 15. Changes in shape would cause errors in multivibrator firing time.

In order to compensate for this a control is provided for selecting the repetition rate being used. This control adjusts the bias on the multivibrator grid. Figure 7.6 shows how adjusting the bias can compensate for changing wave shapes.

The signals are drawn with equal peak amplitudes because the automatic pulse-height circuit accomplishes that (Nos. 21, 16, etc.). Since the critical firing voltage of the multivibrator may vary with aging of the multivibrator tubes, the bias adjustments are made with reference to the actual critical firing voltage. The method is described in the next section. Time delays in the discriminator circuit also are compensated by bias of the multivibrator.

"Slide" Control, Block 19

The voltage required to trigger the multivibrator may vary as the multivibrator tubes age. This would have serious consequences since with a sloping leading edge on the triggering pulse (at the output of No. 10), changes in critical firing voltage would cause changes in firing time. In order to eliminate this error source, the critical triggering voltage of the multivibrator may be calibrated each time the circuit is used. The bias on the multivibrator grid is adjusted with the firing voltage as a reference, not with ground voltage as a reference. This is accomplished by means of a "slide" control. The multivibrator grid is biased highly negative. Then the amount of negative bias is manually decreased slowly until the multivibrator just starts firing. A microammeter indicates when multivibrator triggering starts. The slide control is then locked in the position at which firing just starts. Further changes in bias use this firing voltage as a reference point.

Multivibrator, Block 15

The multivibrator is of the so-called "one shot" type. It is nonsymmetrical in order to achieve maximum steepness of the leading edge of the square wave. Because it is asymmetrical, an overshoot occurs when the multivibrator flops back to its normal state. It was necessary to suppress that overshoot; if not suppressed, it would produce a second positive pulse upon differentiation. The overshoot was suppressed by the addition of a diode between grid and cathode of the second multivibrator tube. That gives the grid circuit of that tube a "unilateral" time constant. The time constant is different for a discharge of one polarity than it is for a discharge of the opposite polarity.

Notice that two a-c transients are produced by the mixer and filter, Nos. 7 and 8. The first of these is used; the second must be suppressed. At the suggestion of Sutton, the second pulse is suppressed by disabling the circuit. The circuit is disabled at the time the second pulse arrives. At first, another method of image rejection was being considered, in which the second pulse was to be eliminated by a high-pass filter preceding the first amplifier, Block 2. This would have required switching in order to place a sharp cut-off point between the frequencies of desired triggering frequency and dee-oscillator frequency at time of image production. Such a filter was designed, the filter being treated as a continuously variable m-derived type. That alternative was abandoned in view of the susceptibility of high-frequency switching circuits to unforeseen difficulties.

The image pulse (which is produced at the time the dee-oscillator frequency is about 1 megacycle lower than the local oscillator frequency) is suppressed in the multivibrator. This is accomplished by having the multivibrator square wave persist until after the image pulse. Both the desired pulse and the image pulse reach the multivibrator; however, the multivibrator is designed so as to be still cycling at the time of receipt of the image pulse. The spacing in microseconds between the desired pulse and the image pulse varies for different repetition rates. If the length of the square wave produced by the multivibrator were to be made long enough to overlap the image pulse at the lowest repetition rate used, then the square-wave length would be too long for foolproof operation when the cyclotron is operated at its maximum repetition rate. Consequently the length of the multivibrator square wave in microseconds must be changed for various repetition rates by switching.

Differentiation and Suppression

The multivibrator produces a positive square wave. (The leading edge of the multivibrator square wave occurs at the time that the dee-oscillator frequency reaches the desired triggering frequency.) The whole square wave is differentiated with respect to time by an RC circuit. This is done in order that the output wave form of the complete discriminator may be a narrow pulse produced at exactly the instant at which the deflector is to be fired. Upon differentiation, the leading edge of the multivibrator square-wave output produces a positive pulse. The trailing edge produces an undesired negative pulse. The negative pulse is suppressed by a diode. The positive pulse is transformed (in impedance) by the output stages to an impedance suitable for matching a transmission line.

Output Amplifiers, Block 17

Multiple output circuits are provided. The reason for using separate output tubes is that equipment of an unknown nature will be connected to one of the outputs. If the equipment connected to one output has a large shunt capacity (or otherwise low impedance), it would attenuate the pulse for the deflector equipment. In order to avoid that, a dual triode was used as an output tube with its grids connected in parallel. Separate cathode-follower output circuits are thus provided. One of these outputs will be permanently connected to the deflector equipment; the other may be used for experimental purposes. Both outputs are 73 ohms. The output signal is 8 volts high (peak), 1.2 μ sec long, and has a rise time of 0.2 μ sec.

Oven Construction

An oven was designed and constructed to house some temperature-critical elements. The crystals which control the local oscillators, the oscillator tubes themselves, and the 1-megacycle variable-frequency filter of the 6L7 plate circuit are all enclosed in the oven. The oven walls are a sandwich of sheet metal and asbestos board. The asbestos board is between two sheets of cadmium-plated steel thus providing a wall with low thermal conductance. Because of the softness of the asbestos board, it could not be tapped in order to mount the sheet metal to it. Bolts could not pass completely through from one piece of sheet metal to the other because of their conductance. For these reasons, Micarta strips were used as a framework for the walls. The visible material is Micarta but the majority of the wall sandwich is filled with asbestos board. The thermal conductivity of asbestos board is one-fifth that of free air.

The oven is designed in such a way that the back may be easily removed first, when the oven circuits must be serviced. The top of the oven is most easily removed second. This might some day be necessary in order to replace one of the large variable condensers. Care was taken in the design of the oven and in the location of the components in it, in order to simplify maintenance. By the removal of a panel accessible from the rear of the equipment, it is possible to replace oscillator tubes and to change the range of frequencies over which the equipment will operate. Changing the range of frequencies requires the changing of crystals, but they are easily accessible. If it is desired to make a small change in the range of frequencies over which the equipment will operate, it is necessary only to replace the crystals and readjust some variable condensers. If the range change is very great, say in the neighborhood of 30 megacycles, it is necessary to change oscillator transformers. They are plug-in units. The coils are wound on the inside of a coil form instead of the outside. In this way the inductance cannot be altered by rough handling in changing the coils.

It is necessary to make final adjustments on the oscillators while the oscillators are at their operating temperature. That means the oscillator adjustments must be accessible from outside the oven; but they are located inside. They are made accessible by providing holes in the back panel through which an adjusting screw driver may be inserted. A fuse body is used in this hole to provide a tube for guiding the screw driver to the adjusting screw. Also the use of a fuse body provides a convenient cap for the hole when not in use. There is a coarse adjustment on the oscillators which must be made with the back panel off. However, all adjustments which are temperature critical may be made with the oven completely closed.

The thermostat regulates the temperature at 140° F. There are two 125-watt heaters connected in series when the thermostat contacts are closed, dissipating 62½ watts of heat power. A thermometer on the front panel shows the oven temperature. Felt seals are used where walls are joined and also around control shafts which must project through the oven. All materials used in the oven (insulation, etc.) were subjected to 140° F for 17 hr and showed no ill effects immediately or after cooling.

Adjusting Instructions

The plate circuit of each local oscillator should be tuned to a frequency slightly above its crystal frequency. When so tuned, it presents an inductive reactance to the oscillator tube, a condition necessary for oscillation. The adjusting condensers may be tuned while observing the amplitude of oscillations at the buffer-tube output. Figure 7.7 shows the best setting.

7.4 SPECIFICATIONS

Input voltage (from dee oscillator), 0.003 volt rms
 Input impedance, 50 ohms
 Output voltage (a pulse)
 Peak voltage, 7 volts
 Rise time, 0.2 μ sec
 Duration, 1.2 μ sec
 Output impedance, 73 ohms
 Power requirements
 110-volt, 60-cycle line
 Power consumption $\approx$ 80 watts when oven heaters are off
 Power consumption $\approx$ 240 watts when oven heaters are on
 Tube complement, 27 tubes of available types
 Probable error in firing time, <1 μ sec
 Range of firing frequency, 19 to 21½ megacycles, with provision for changing this

7.5 OPERATING INSTRUCTIONS

Turn on the power switch.
 Wait 15 sec, then turn on plate switch.
 Allow several minutes for equipment to warm up.

To Frequency-Calibrate the Filter, Block 8

1. Set "calibrate-operate" switch to "calibrate."
2. Set "repetition-rate" switch to any position other than "slide."
3. Set the "range" switch to the fifth lowest range; set "triggering-frequency" control to zero; adjust "pad" control to the position of maximum meter deflection.
4. Set "range" switch to sixth lowest range; set "triggering-frequency" control to 0.25 megacycles; adjust "trim" control to the position of maximum meter deflection.
5. Repeat steps 3 and 4 until no change is evident; then lock the "pad" and "trim" controls.

To Calibrate the Multivibrator Firing Voltage

The dee oscillator must be operating to perform this calibration.

1. Set "calibrate-operate" switch to "operate."
2. Set "repetition-rate" switch to "slide."
3. Turn the "slide" control fully counterclockwise, then rotate it slowly clockwise until the multivibrator barely starts firing, as indicated by a meter deflection.
4. Lock the "slide" control in this position.

To Operate the Discriminator

1. Set "calibrate-operate" switch to "operate."
2. Set "repetition-rate" switch to the rate of frequency modulation of the dee oscillator.
3. Select the desired triggering frequency on the "range" switch and "triggering-frequency" control. Their total is the triggering frequency.

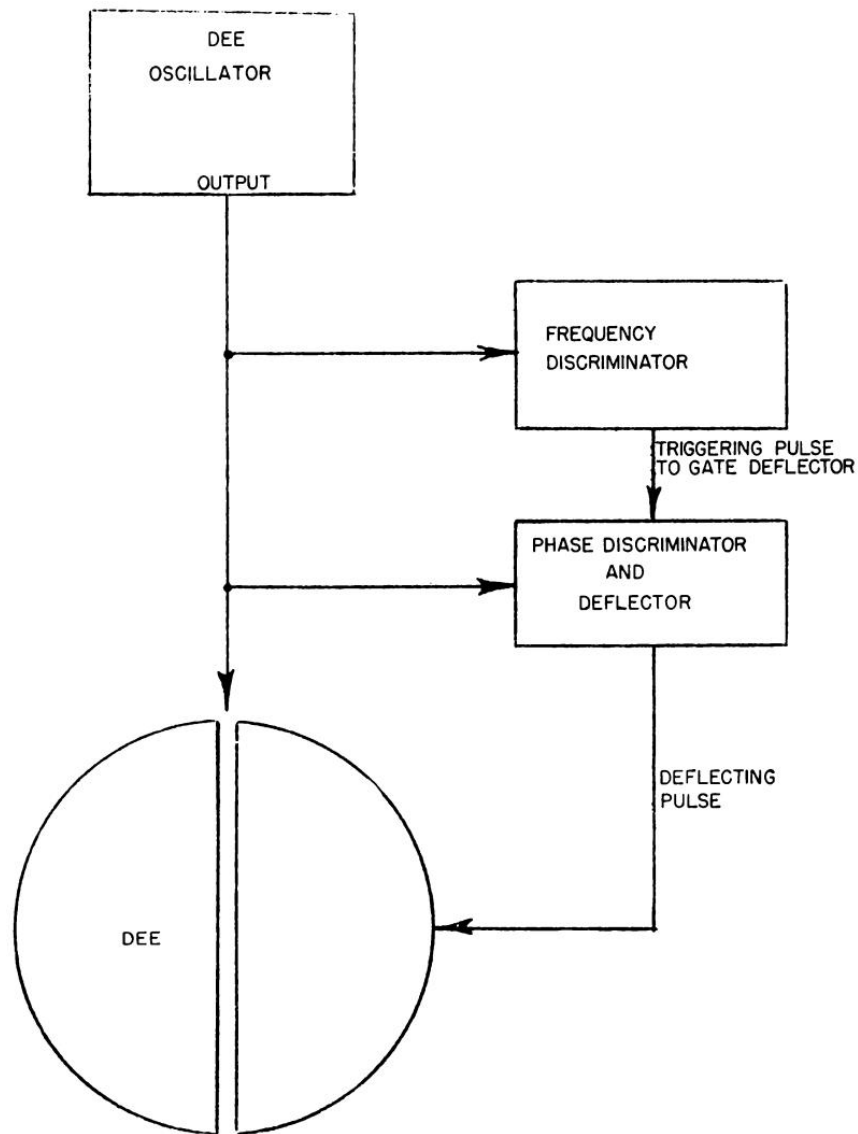


Fig. 7.1—Deflection system.

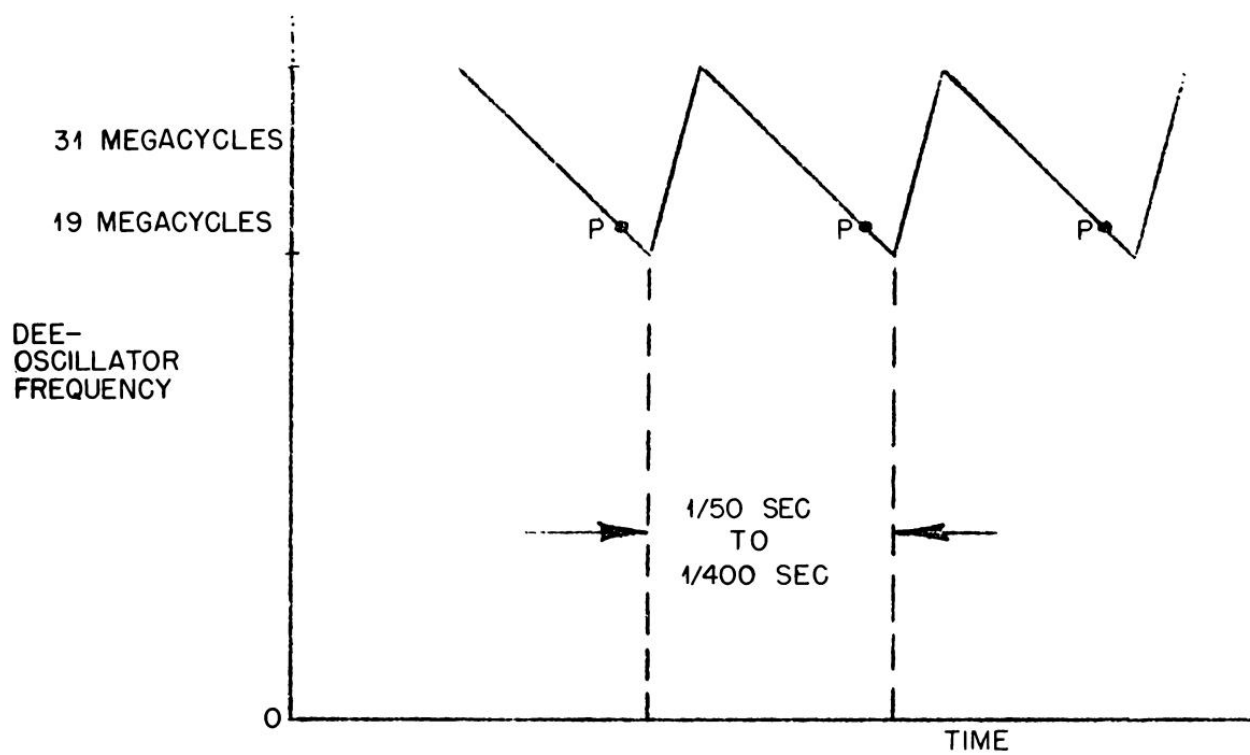


Fig. 7.2—Dee-oscillator output.

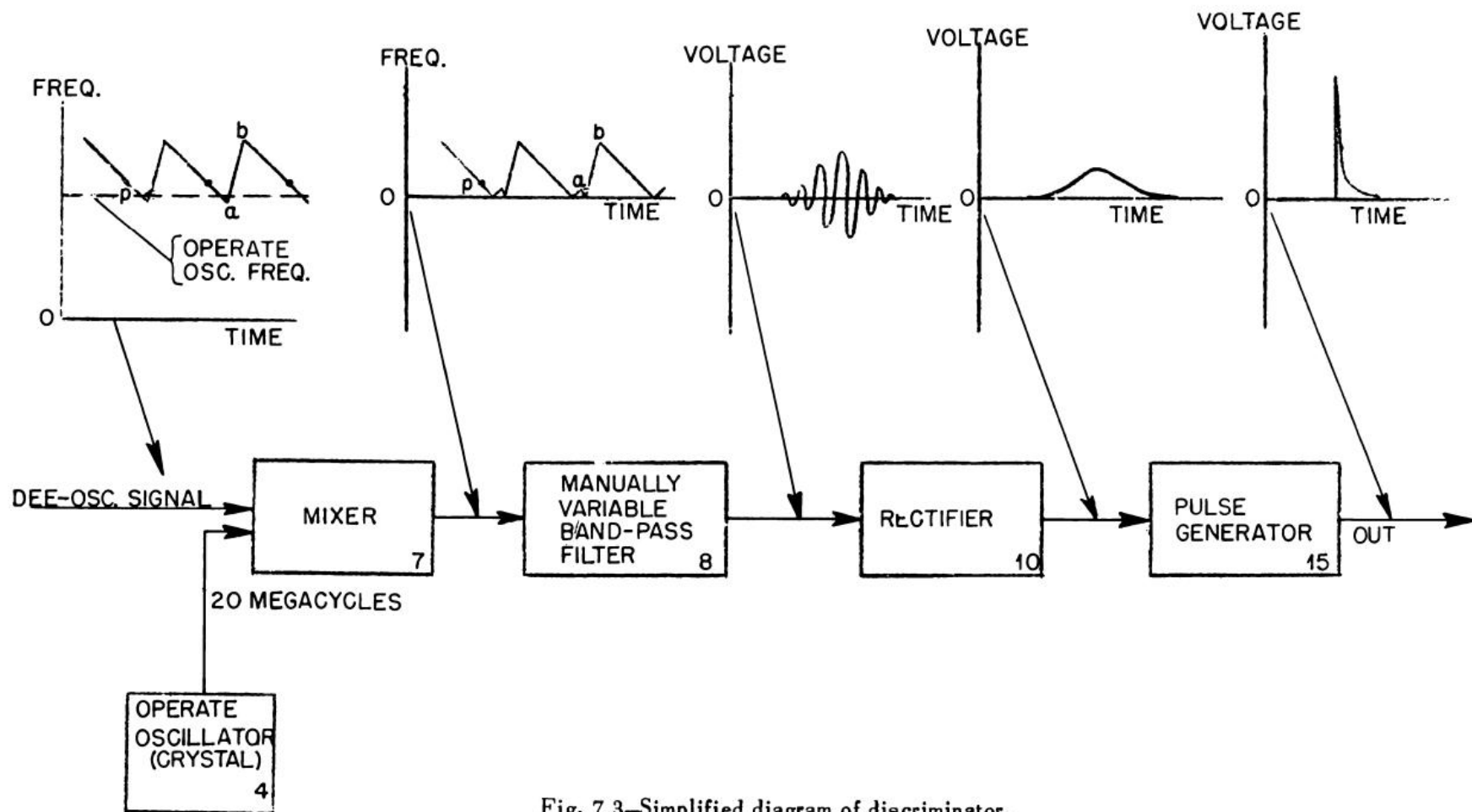


Fig. 7.3—Simplified diagram of discriminator.

Fig. 7.4—See tip-in page 181.

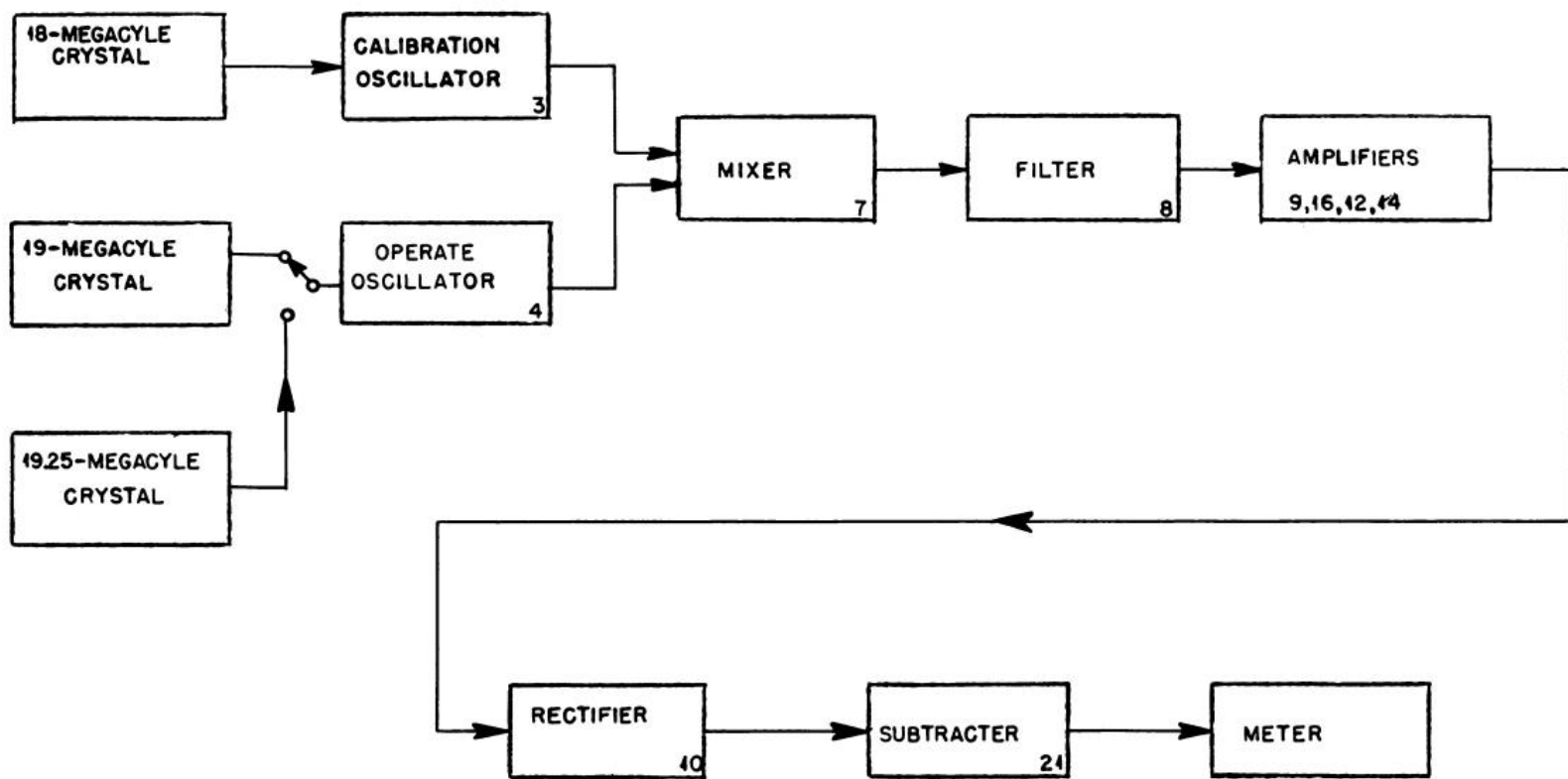


Fig. 7.5—Frequency calibration of the filter, Block 8.

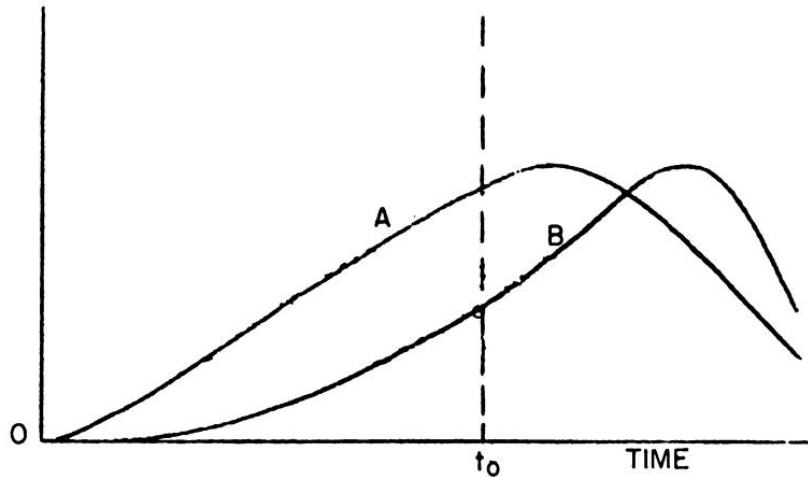


Fig. 7.6a—Bias compensation for change of repetition rate. Signal voltage at multivibrator grid, bias omitted. t_0 (desired time of deflection) is the time when the mixer output is at the center frequency of filter No. 8 of Figs. 7.4 and 7.5. A, repetition rate of 50 counts/sec; B, repetition rate of 400 counts/sec.

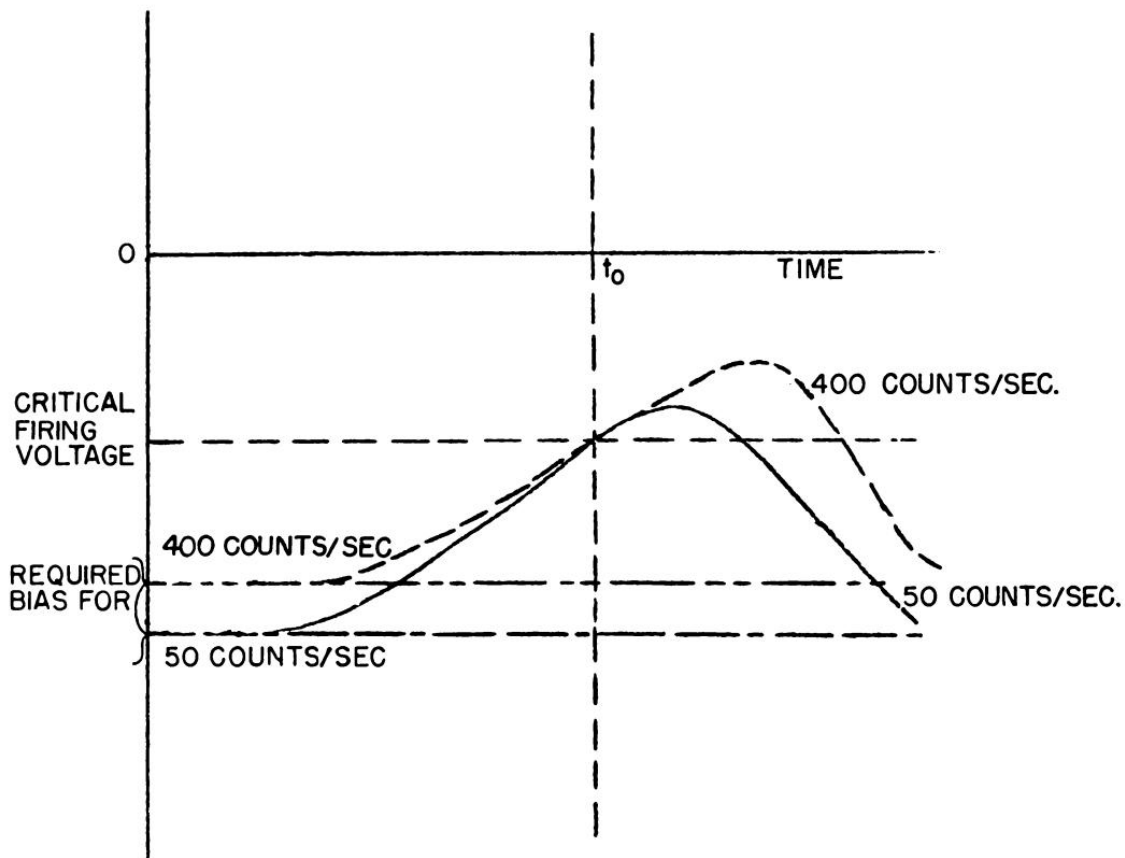


Fig. 7.6b—Bias compensation for change of repetition rate. Signal voltage at multivibrator grid, including bias. t_0 (desired time of deflection) is the time when the mixer output is at the center frequency of filter No. 8 of Figs. 7.4 and 7.5.

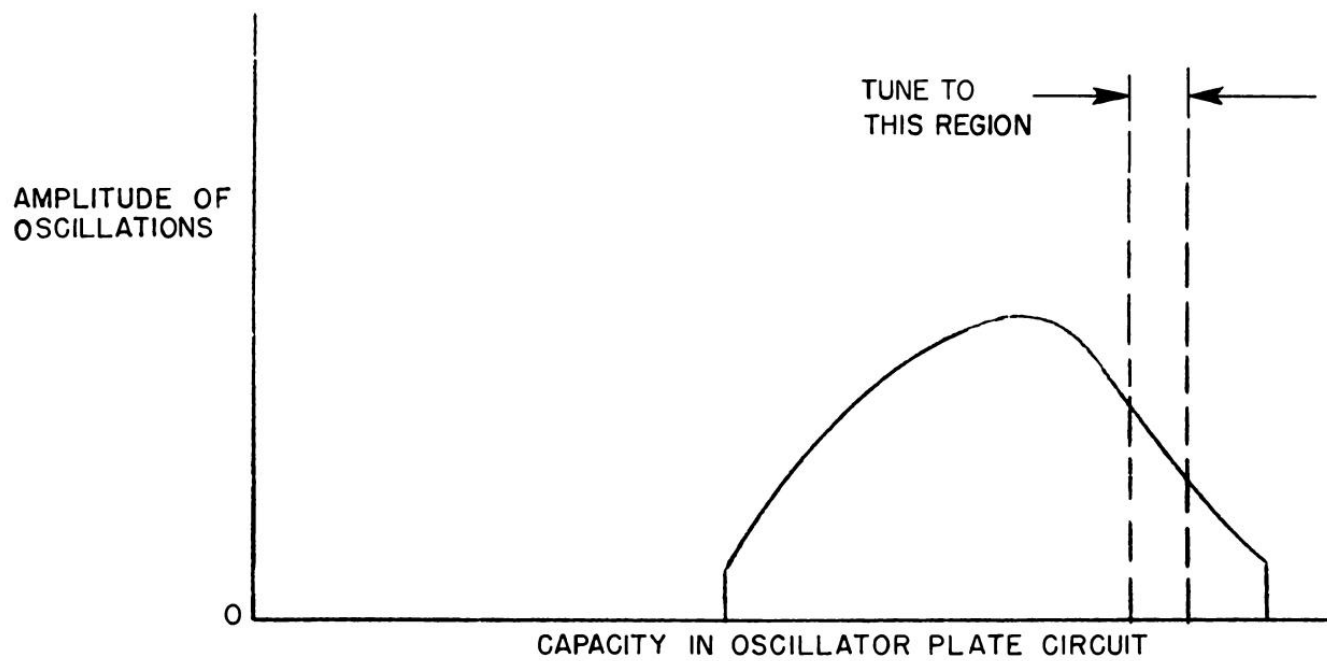
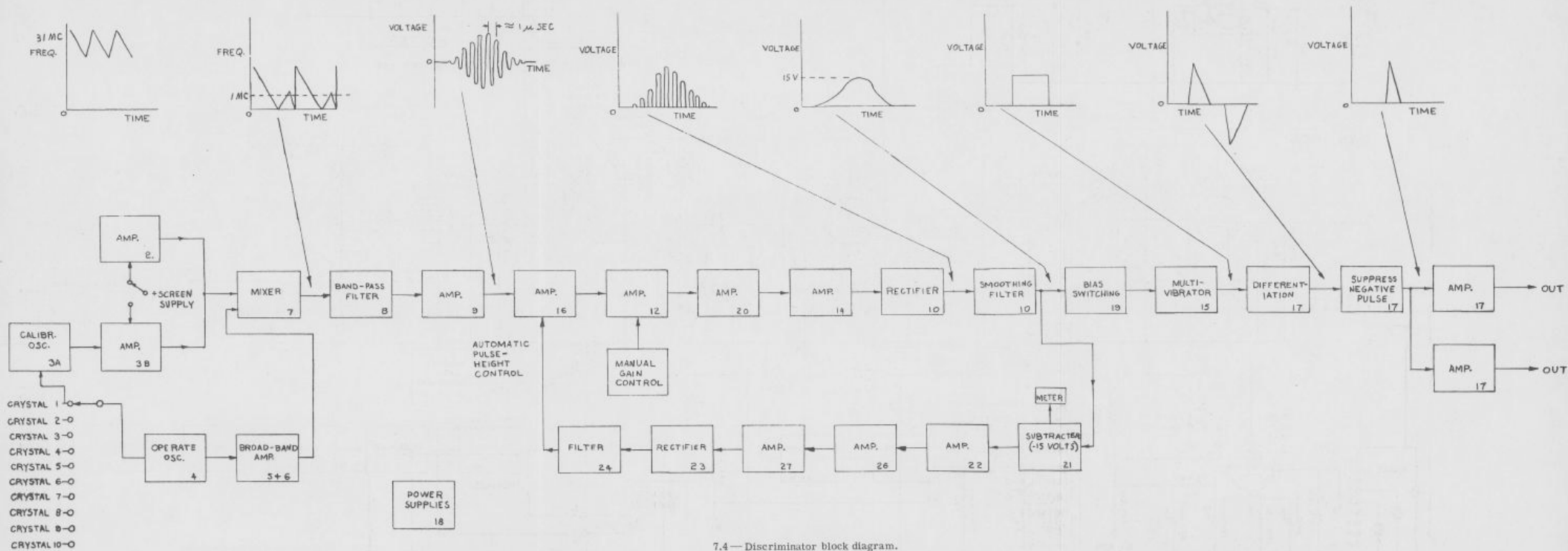
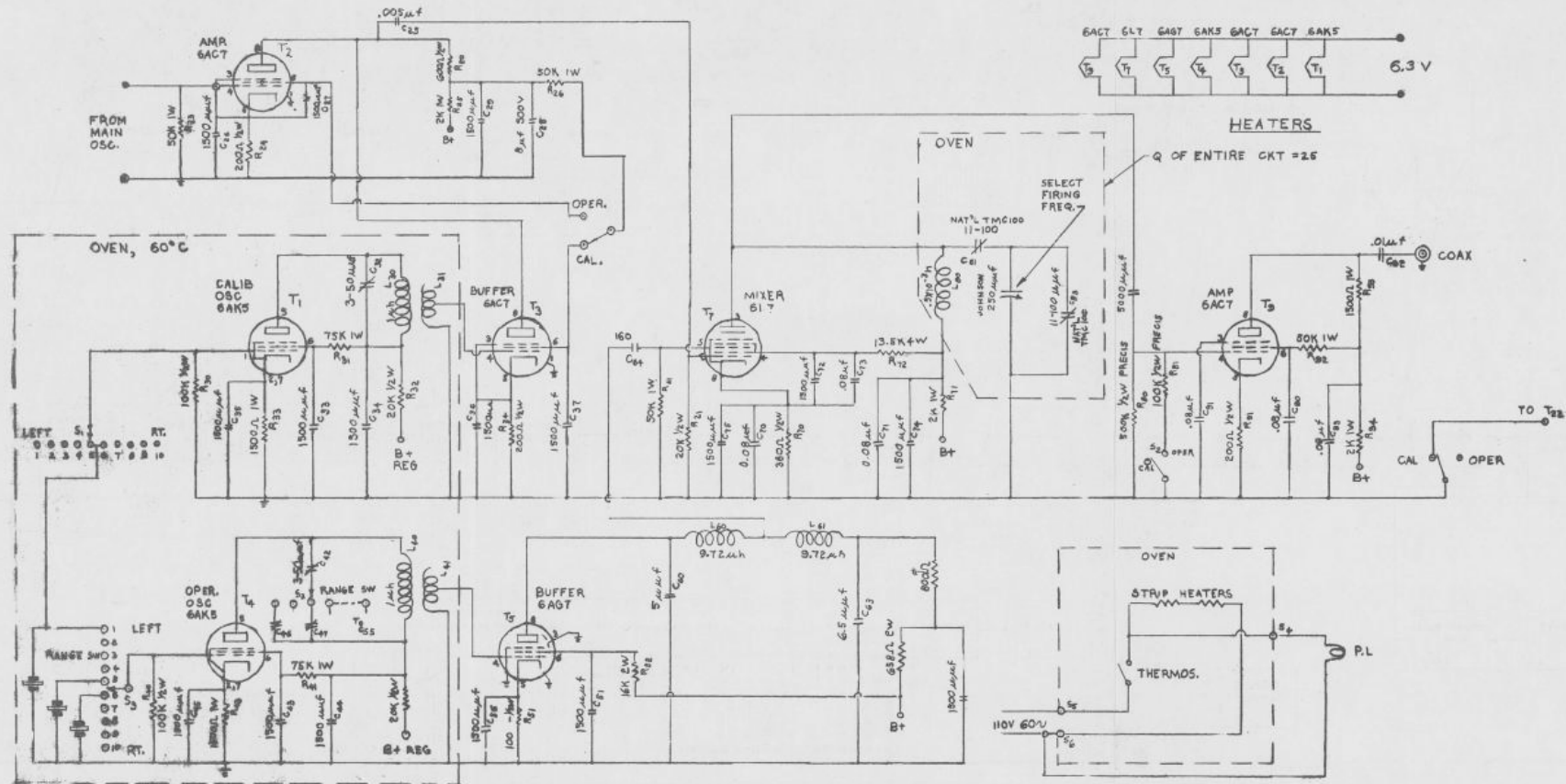


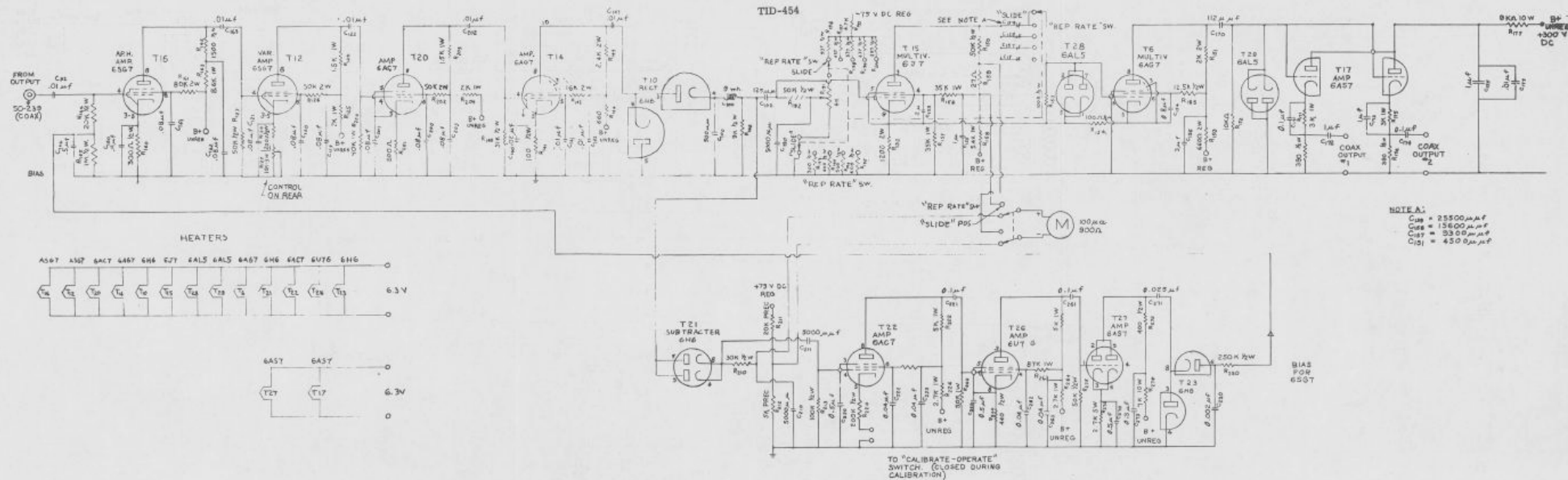
Fig. 7.7—Oscillator adjustment.



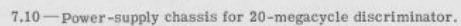
7.4—Discriminator block diagram.



7.8—Input section of 20-megacycle discriminator.



7.9—Output section of 20-megacycle discriminator.



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